GSC Advanced Research and Reviews

eISSN: 2582-4597 CODEN (USA): GARRC2

Cross Ref DOI: 10.30574/gscarr

Journal homepage: https://gsconlinepress.com/journals/gscarr/

(RESEARCH ARTICLE)

Check for updates

A new class of orthogonal polynomials as trial function for the derivation of numerical integrators

F. L. Joseph *

Department of Mathematical Sciences, Bingham University Karu, Nigeria.

GSC Advanced Research and Reviews, 2022, 12(03), 001–012

Publication history: Received on 26 July 2022; revised on 30 August 2022; accepted on 01 September 2022

Article DOI: https://doi.org/10.30574/gscarr.2022.12.3.0229

Abstract

This paper presents a set of newly constructed polynomials valid in interval [-1, 1] with respect to weight function w(x) = x + 1. For applicability sake, the polynomials shall be employed as trial function to develop a fast, efficient and reliable block algorithm for the numerical solution of ordinary differential equations with application to second order initial value problems. Collocation and interpolation techniques were adopted for the formulation of self-starting continuous hybrid schemes. Findings from the analysis of the basic properties of the method using appropriate existing theorems show that the developed schemes are consistent, zero-stable and hence convergent. On implementation, the superiority of the scheme over the existing method is established numerically. Further investigation of the properties of these polynomials is ongoing as we hope to discuss this in the future paper.

Keywords: Collocation; Interpolation; Orthogonal polynomials; Block method; Hybrid

1 Introduction

Polynomials are useful in a wide variety of fields, including biology, economics, cryptography, chemistry, coding, engineering and advanced mathematical fields, such as numerical analysis to mention a few. The utility of polynomial functions for modeling remains as physical and real-world phenomena are modelled through these.

The problem of approximating a function is of great significance in numerical analysis due to its importance in the development of software for digital computers. Polynomials such as Chebyshev, Legendre, Hermite, Laguerre etc. have been widely used in this respect. The orthogonal polynomials have its origin in the nineteenth century theories of continued fractions and the moment problem. Classical orthogonal polynomials have found widespread use in all areas of science and engineering. Typically, they are used as trial functions to expand other more complicated functions in which many at times, arise from initial or boundary value problems.

The focus here is to construct a set of polynomials and use them to develop continuous method in order to establish their application in solving the class of initial value problems of the form

$$y^{m}(x) = f(x, y, y^{'}, \ldots, y^{m-1}) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$
$$y^{r}(x_0) = y_r, r = 0,1,\ldots, k-1$$

Since most of these equations are difficult to solve, efficient ODEs solvers are much needed to approximate them. The numerical solution of ODEs by collocation methods has been studied (see [1], [2], [3], [4], [5], [6], [7]). In the recent

* Corresponding author: F. L. Joseph
Department of Mathematical Sciences, Bingham University Karu, Nigeria.

years, the solutions of (1) for cases m = 1, 2 & 3 have been extremely discussed (see [8], [9], [10], [11], [12]). The approach of reducing the higher order of (1) to a system of first order and the consequent setback has also been discussed in [13].

Predictor-corrector method was later adopted and applied but has its setbacks which were discussed in [13], [14] and [15]. To cater for the setback of predictor-corrector methods, the approach of block method came into being. Milne in [16] proposed a method called block method as a means of obtaining starting values which [17] and [18] developed into algorithms for general use. Later, the modified self-starting block method was given in [15] and [19] as

$$Y_m = ey_n + h^{\mu-\lambda} df(y_n) + h^{\mu-\lambda} bF(y_m) \quad \ldots\ldots\ldots\ldots\ldots\ldots. (2)$$

Where;

e is s x s vector,
d is r-vector
b is r x r vector,
s is the interpolation points
r is the collocation points.
F is a k-vector whose jth entry is $f_{n+j} = f(t_{n+j}, y_{n+j})$
μ is the order of the differential equation.

Given a predictor equation in the form

$$Y_m^{(0)} = ey_n + h^{\mu} df(y_n) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

Putting (3) in (2) gives

$$Y_m = ey_n + h^{\mu} df(y_n) + h^{\mu} bF(ey_n + h^{\mu} dfy_n) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

which is called a self-starting block-predictor-corrector method, [20].

The block (4) is a simultaneous producing approximations to the solution of (1) at a block of desired points. However, the effectiveness of these ODE solvers depends on the types of trial functions used in developing the schemes. Various trial functions such as, the Chebyshev polynomials which was introduced in [14] as basis function for the solution of linear differential equations in term of finite expansion, the Legendre polynomials, Power series and the Canonical polynomials have been used to derive continuous schemes.

In what follows, we shall construct a set of polynomials valid in interval [-1, 1] with respect to weight function w(x) = x + 1.

1.1 Construction of Orthogonal Basis Function

Let the function $q_n(x)$ be defined as

$$q_r(x) = \sum_{r=0}^{n} C_r^{(n)} x^r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

Where; $q_r(x)$ satisfies

$$< q_m(x), q_n(x) > = 0, \quad m \neq n, [-1,1] \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

For the purpose of constructing the basis function, we use additional property that

$$q_n(1) = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

For r = 0 in (5),

$$q_0(x) = C_0^{(0)}$$

From (7),

$$q_0(1) = C_0^{(0)} = 1$$

Hence,

$$q_0(x) = 1$$

For r = 1 in (5),

$$q_1(x) = C_0^{(1)} + C_1^{(1)} x \ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

By definition (7), (8) gives

$$C_0^{(1)} + C_1^{(1)} = 1 \ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

And

$$< q_0, q_1 > = \int_{-1}^{1} (x+1)\, q_0(x) q_1(x) dx$$

Which implies

$$C_1^{(1)} + 3C_0^{(1)} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

Solving (9) and (10) and substituting the outcomes into (8), we have

$$q_1(x) = \frac{1}{2}(3x - 1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

When r = 2 in (5),

$$q_2(x) = C_0^{(2)} + C_1^{(2)} x + C_2^{(2)} x^2 \ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

By definition (7), (12) gives

$$C_0^{(2)} + C_1^{(2)} + C_2^{(2)} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

$$< q_0, q_2 > = \int_{-1}^{1} (x+1)\, q_0(x) q_2(x) dx$$
$$= 0$$

Which implies

$$2C_0^{(2)} + \frac{2}{3}C_1^{(2)} + \frac{2}{3}C_2^{(2)} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

$$< q_1, q_2 > = \int_{-1}^{1} (x+1)\, q_1(x) q_2(x) dx$$
$$= 0$$

Which gives

$$\frac{2}{3}C_1^{(2)} + \frac{4}{15}C_2^{(2)} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

Solving (13), (14), (15) and substituting the resulting values into (12), we have

$$q_2(x) = \frac{1}{2}(5x^2 - 2x - 1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

When r = 3 in (5),

$$q_3(x) = C_0^{(3)} + C_1^{(3)}x + C_2^{(3)}x^2 + C_3^{(3)}x^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

By definition (7), (17) gives

$$C_0^{(3)} + C_1^{(3)} + C_2^{(3)} + C_3^{(3)} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18)$$

$$< q_0, q_3 > = \int_{-1}^{1} (x+1)\, q_0(x) q_3(x) dx$$
$$= 0$$

Which implies

$$2C_0^{(3)} + \frac{2}{3}C_1^{(3)} + \frac{2}{3}C_2^{(3)} + \frac{2}{5}C_3^{(3)} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19)$$

$$< q_1, q_3 > = \int_{-1}^{1} (x+1)\, q_1(x) q_3(x) dx$$
$$= 0$$

This leads to

$$\frac{2}{3}C_1^{(3)} + \frac{4}{15}C_2^{(3)} + \frac{2}{5}C_3^{(3)} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (20a)$$

$$< q_2, q_3 > = \int_{-1}^{1} (x+1) q_2(x) q_3(x) dx$$

$$= 0$$

$$\Rightarrow \frac{4}{15}C_2^{(3)} + \frac{4}{35}C_3^{(3)} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (20b)$$

Solving (18) – (20) and substituting the resulting values into (17), we obtain

$$q_3(x) = \frac{1}{8}(35x^3 - 15x^2 - 15x + 3) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (21)$$

In the same vein, $q_n(x), n \geq 4$ are developed.

The next seven polynomials are listed below.

$$q_4(x) = \frac{1}{8}(63x^4 - 28x^3 - 42x^2 + 12x + 3)$$

$$q_5(x) = \frac{1}{16}(231x^5 - 105x^4 - 210x^3 + 70x^2 + 35x - 5)$$

$$q_6(x) = \frac{1}{16}(429x^6 - 198x^5 - 495x^4 + 180x^3 + 135x^2 - 30x - 5)$$

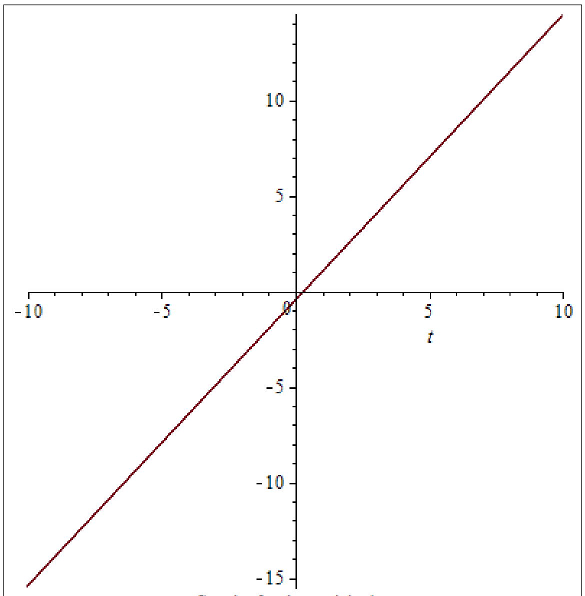
$$q_7(x) = \frac{1}{128}(6435x^7 - 3003x^6 - 9009x^5 + 3465x^4 + 3465x^3 - 945x^2 - 315x + 35)$$

$$q_8(x) = \frac{1}{128}(12155x^8 - 5720x^7 - 20020x^6 + 8008x^5 + 10010x^4 - 3080x^3 - 1540x^2 + 280x + 35)$$

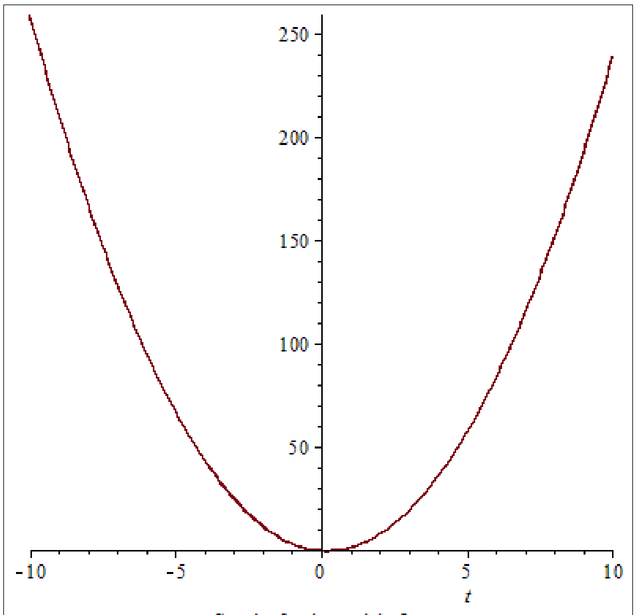
$$q_9(x) = \frac{1}{256}(46189x^9 - 21879x^8 - 87516x^7 + 36036x^6 + 54054x^5 - 18018x^4 - 12012x^3 + 2772x^2 + 693x - 63)$$

$$q_{10}(x) = \frac{1}{256}(88179x^{10} - 41990x^9 - 188955x^8 + 79560x^7 + 139230x^6 - 49140x^5 - 40950x^4 + 10920x^3 + 4095x^2 - 630x - 63)$$

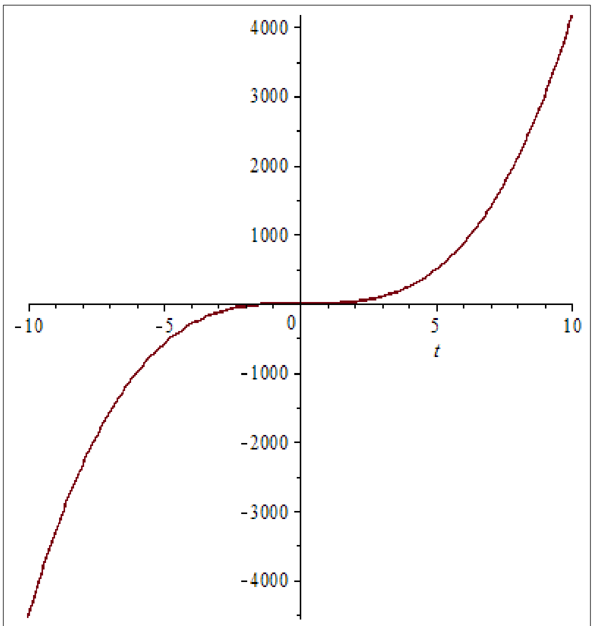
The graphs of these polynomials are shown hereunder shown as further investigation of their properties is ongoing, which shall be discussed in the future paper.



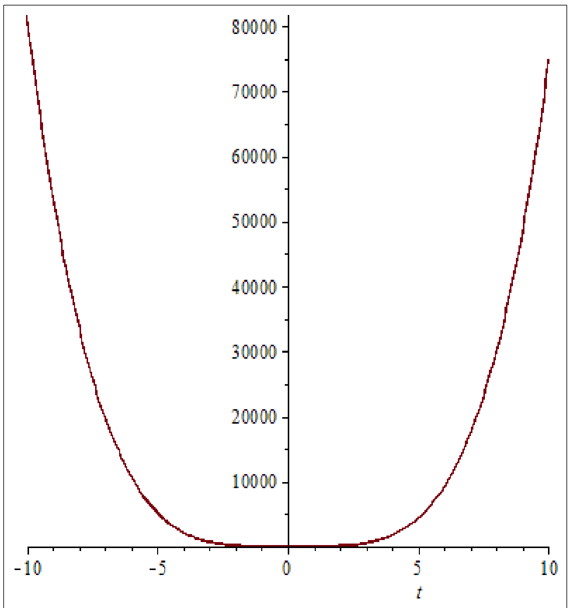
Graph of polynomial q1



Graph of polynomial q2



Graph of polynomial q3



Graph of polynomial q4

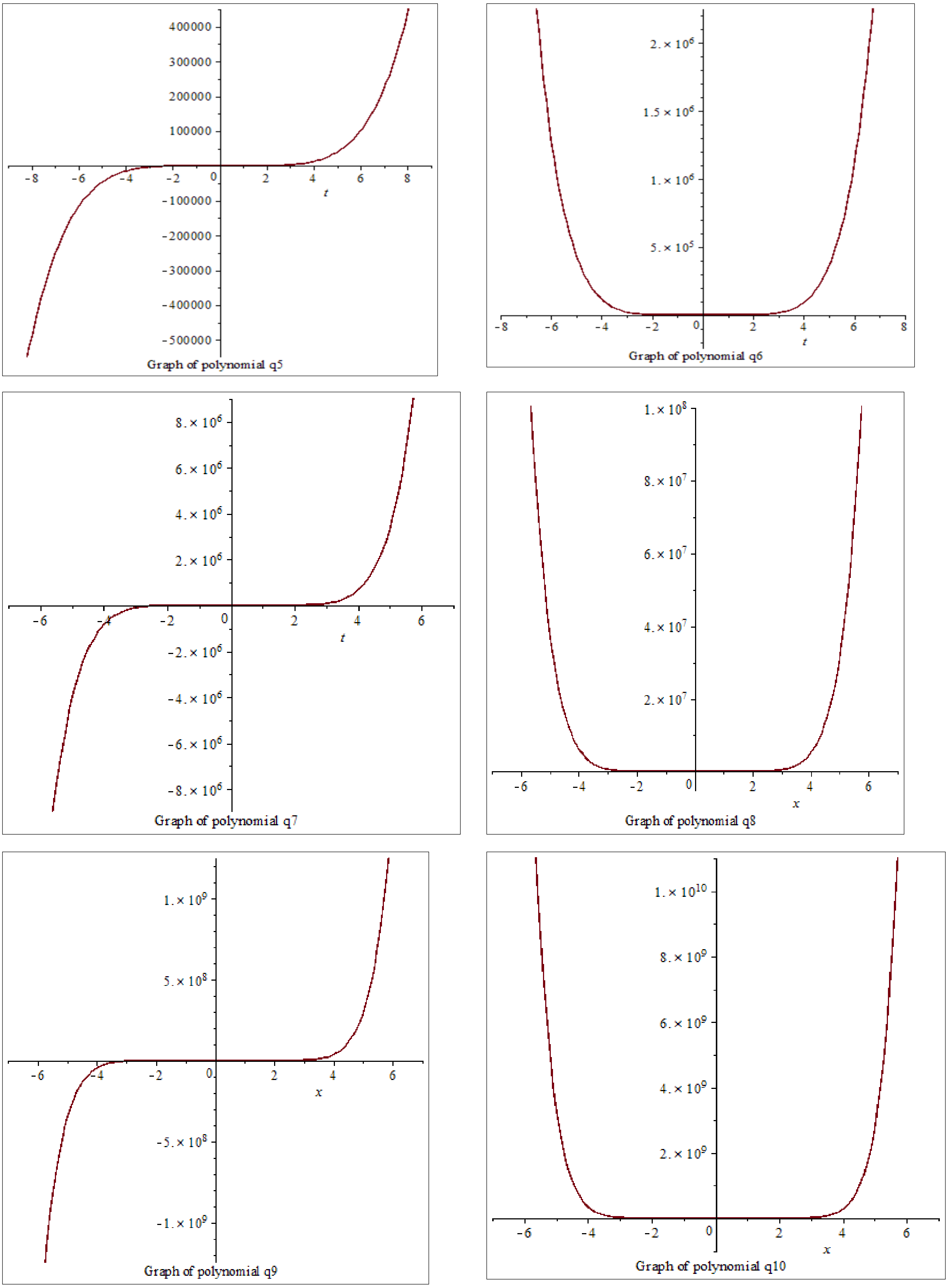


Figure 1 Graph of Polynomials

In the next section, the applicability of these polynomials (as trial functions) shall be investigated using a well-known method and in section four, these polynomials we be employed as trial function to develop a numerical scheme.

1.2 Literature Review

To investigate the applicability of the derived orthogonal polynomials, we briefly review here the work of Adam-Moulton on derivation of three-step implicit method whose discrete scheme is

$$y_{n+3} = y_{n+2} + \frac{h}{24}(f_n - 5f_{n+1} + 19f_{n+2} + 9f_{n+3})$$

For this purpose, we shall seek an approximation of the form

$$y(x) = \sum_{r=0}^{s+k-1} a_r q_r(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

Where; $q_r(x)$ is the orthogonal polynomials derived.

We collocate and interpolate (22) at $x = x_{n+i}, i = 0(1)3$ and $x = x_{n+2}$ respectively to obtain a system of equations which are solved and the resulting values of a_r are substituted back into (22) to have a continuous schemes. Evaluating the continuous scheme at the grid point $x = x_{n+3}$ yields the Adams-Moulton explicit three-step method.

We shall now employ the set of polynomials to formulate a continuous scheme through which numerical solutions of initial value problems in ordinary differential equations are obtained. We hope that these newly generated polynomials will stimulate further interest which will lead to a thorough investigation of the new class of the polynomials.

2 Formulation of Numerical Integrator

In this section, our objective is to derive a two-step continuous hybrid linear multistep method in the sub-interval [x_n, x_{n+p}] of [a, b] where $x = \frac{2X - 2x_n - ph}{ph}$ and p varies as the method to be derived. For this case, p = 2.

The procedure involves interpolating (22) at s = $\frac{1}{3}$ & $\frac{2}{3}$ and collocating the second derivative of (22) at k = 0, $\frac{1}{3}$, $\frac{2}{3}$, 1 and 2. The a_r, r = 0(1)6 from the resulting system of equations are obtained and substituted into (22) to have the continuous equation

$$y(x) = \alpha_{\frac{1}{3}}(x) y_{n+\frac{1}{3}} + \alpha_{\frac{2}{3}}(x) y_{n+\frac{2}{3}} + h^2(\beta_0(x) f_n + \beta_{\frac{1}{3}}(x) f_{n+\frac{1}{3}} + \beta_{\frac{2}{3}}(x) f_{n+\frac{2}{3}} + \beta_1(x) f_{n+1} + \beta_2(x) f_{n+2}) \quad (23)$$

Evaluating equation (23) at x = x_{n+2} yields our main method as

$$y_{n+2} = 5y_{n+\frac{2}{3}} - 4y_{n+\frac{1}{3}} + \frac{h^2}{3240}(-490 f_n + 2388 f_{n+\frac{1}{3}} - 2715 f_{n+\frac{2}{3}} + 4220 f_{n+1} + 197 f_{n+2}) \ldots\ldots\ldots .(24)$$

To develop the block method from the continuous scheme, we adopt the general block formula proposed in (Awoyemi et al. 2011) in the normalized form given as

$$A^{(0)} Y_m = e y_m + h^{\mu-\lambda} d f(y_m) + h^{\mu-\lambda} b F(y_m) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (25)$$

Evaluating the first derivative of (23) at x = x_{n+j}, j = 0, $\frac{1}{3}$, $\frac{2}{3}$, 1, 2, substituting the resulting equations and the main method into (25) and solving simultaneously gives a block formulae represented as

$$\left.\begin{aligned}
y_{n+\frac{1}{3}} &= y_n + \frac{h}{3}y'_n + \frac{h^2}{64800}(1870f_n + 2532f_{n+\frac{1}{3}} - 1095f_{n+\frac{2}{3}} + 300f_{n+1} - 7f_{n+2}) \\
y_{n+\frac{2}{3}} &= y_n + \frac{2h}{3}y'_n + \frac{h^2}{4050}(270f_n + 696f_{n+\frac{1}{3}} - 105f_{n+\frac{2}{3}} + 40f_{n+1} - f_{n+2}) \\
y_{n+1} &= y_n + hy'_n + \frac{h^2}{2400}(250f_n + 756f_{n+\frac{1}{3}} + 135f_{n+\frac{2}{3}} + 60f_{n+1} - f_{n+2}) \\
y_{n+2} &= y_n + 2hy'_n + \frac{h^2}{750}(50f_n + 1080f_{n+\frac{1}{3}} - 675f_{n+\frac{2}{3}} + 1000f_{n+1} + 45f_{n+2}) \\
y'_{n+\frac{1}{3}} &= y'_n + \frac{h}{32400}(3860f_n + 9234f_{n+\frac{1}{3}} - 3105f_{n+\frac{2}{3}} + 830f_{n+1} - 19f_{n+2}) \\
y'_{n+\frac{2}{3}} &= y'_n + \frac{h}{4050}(440f_n + 1836f_{n+\frac{1}{3}} + 405f_{n+\frac{2}{3}} + 20f_{n+1} - f_{n+2}) \\
y'_{n+1} &= y'_n + \frac{h}{1200}(140f_n + 486f_{n+\frac{1}{3}} + 405f_{n+\frac{2}{3}} + 170f_{n+1} - f_{n+2}) \\
y'_{n+2} &= y'_n + \frac{h}{150}(-40f_n + 324f_{n+\frac{1}{3}} - 405f_{n+\frac{2}{3}} + 380f_{n+1} + 41f_{n+2})
\end{aligned}\right\} \dots\dots\dots\dots(26)$$

2.1.1 *Definition* 4.1: Order and Error Constant

The linear operator L of the block (25) is defined as

$$L\{y(x):h\} = Y_m - ey_n + h^{\mu-\lambda}df(y_m) + h^{\mu-\lambda}bF(y_m) \dots\dots\dots\dots\dots\dots\dots\dots (27)$$

Using Taylor series expansion to expand $y(x_n + ih)$ and $f(x_n + jh)$, (27) becomes

$$L\{y(x):h\} = C_0 y(x) + C_1 hy'(x) + C_2 h^2 y''(x) + \dots + C_p h^p y^{(p)}(x) \dots\dots\dots\dots\dots\dots (28)$$

The block (25) and associated linear operator are said to have order p if

$$C_0 = C_1 = \dots = C_p = C_{p+1} = C_{p+2} \neq 0$$

The term $C_{p+2} \neq 0$ is called the error constant and the local truncation error is given

As $t_{n+k} = C_{p+2} h^{(p+2)} y^{(p+2)}(x_n) + 0h^{(p+3)}$

Thus, equations (24) is of order 5 and error constant $-\frac{73}{87480}$. The formulae in the block (26) are all of order 5 with error constants C_{p+2} given as

$$C_{P+3} = \left[\frac{829}{110224800}\ \frac{61}{3444525}\ \frac{13}{453600}\ -\frac{11}{14175}\ \frac{211}{5248800}\ \frac{7}{328050}\ \frac{1}{21600}\ -\frac{1}{450}\right]^T$$

respectively.

2.1.2 *Definition 4:* Zero-Stability

The block (25) is said to be zero-stable if the roots z_s = 1, 2... N of the characteristic polynomial $\rho(z)=\det(zA-E)$, satisfies $|z|\le 1$ and the root $|z|=1$ has multiplicity not exceeding the order of the differential equation. Also, as $h^{\mu}\to 0, \rho(z)=z^{r-\mu}(\lambda-1)^{\mu}$, where μ is the order of the differential equation, $r=\dim(A^{(0)})$.

The proposed method has been investigated to be zero stable.

2.1.3 *Definition 4.3:* Consistency

A numerical method is consistent if the order, $p\ge 1$,

This method is consistent owing to the fact that the order $p\ge 1$.

2.1.4 *Definition 4.4:* Convergence

The necessary and sufficient condition for a numerical method to be convergent is for it to be zero-stable and consistent. According to this definition, the method derived is convergent.

The diagram of region of absolute stability is as shown below.

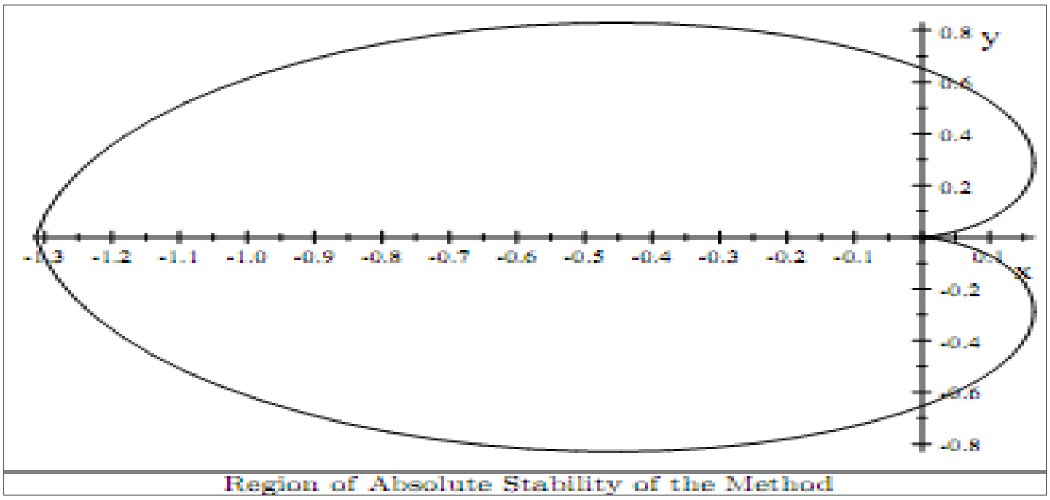


Figure 2 Region of Absolute Stability of the Method

2.1.5 *Numerical Experiment*

We demonstrate the method with the test problem

$$y''-y'=0,\ y(0)=0,\ y'(0)=-1$$

Whose analytical solution is $y(x)=1-\exp(x)$.

Table 1 Table of Results

X	Exact Solution	Approximate Solution	Error	Error in [19]
0.1	−0.10517091807564762480	-0.10517091807239943619	-3.24818861e-12	8.79316E-05
0.2	−0.22140275816016983390	- 0.22140275824581250946	8.564267556 e-11	3.26718E-04
0.3	−0.34985880757600310400	- 0.34985880792001473211	3.4401162811e-10	2.215564E-03
0.4	−0.49182469764127031780	- 0.49182469838377994138	7.4250962358e-10	4.857093E-03

0.5	−0.64872127070012814680	- 0.64872127207862860168	1.37850045488e-09	9.097734E-03
0.6	−0.82211880039050897490	-0.82211880260985294537	2.21934397047e-09	1.4391394E-02
0.7	−1.01375270747047652160	- 1.01375271085798121930	3.3875046977e-09	2.1437918E-02
0.8	−1.22554092849246760460	- 1.22554093333950033000	4.8470327254e-09	2.9898724E-02
0.9	−1.45960311115694966380	-1.45960311790878502900	6.7518353652e-09	4.0300719E-02
1.0	−1.71828182845904523540	- 1.71828183752183259550	9.0627873601e-09	5.255213E-02

3 Conclusion

A successful application of a set of newly constructed polynomials has been demonstrated.

The new polynomials and the formulated scheme are therefore recommended for general purposed use while the further properties shall be discussed in the future paper.

Compliance with ethical standards

Acknowledgments

The Author acknowledged the efforts of the reviewers whose comments have improved the quality of this article.

Conflict of Interest

The author declares that there is no conflict of interest

References

[1] R.B. Adeniyi, E.O. Adeyefa, M.O. Alabi, A Continuous Formulation of Some Classical Initial value Solvers by Non-Perturbed Multistep Collocation Approach using Chebyshev Polynomials as Basis Functions. Journal of the Nigerian Association of Mathematical Physics, 2006, 10: 261 - 274

[2] R.B. Adeniyi, M.O. Alabi, Derivation of Continuous Multistep Methods Using Chebyshev Polynomial Basis Functions, Abacus, 2006, 33(2B): 351 - 361.

[3] A.O. Adesanya, M.O. Udo, A.M. Alkali, A New Block-Predictor Corrector Algorithm for the Solution of $y''' = f(x, y, y', y'')$. American Journal of Computational Mathematics, 2012, 2: 341-344.

[4] E.O. Adeyefa, F.L. Joseph, O.D. Ogwumu, Three-Step Implicit Block Method for Second Order ODEs. International journal of Engineering Science Invention, 2014, 3(2), 34-38.

[5] E.O Adeyefa: A Model for Solving First, Second and Third Order IVPs Directly. Int. J. Appl. Comput. Math, 2021, 7{131), https://doi.org/10.1007/s40819-021-01075-6

[6] E.O Adeyefa and J. O. Kuboye: Derivation of New Numerical Model Capable of Solving Second and Third Order Ordinary Differential Equations Directly, IAENG International Journal of Applied Mathematics, 2020, 50(2), 233-241

[7] S. N. Jator and E. O. Adeyefa. Direct Integration of Fourth Order Initial and Boundary Value Problems using Nystrom Type Methods, IAENG International Journal of Applied Mathematics, 2019, 49(4), 638-649.

[8] E. O. Adeyefa: On formulation of numerical algorithm using newly constructed basis function , Far East Journal of Mathematical Sciences, 2018, 109(1), 201-214, http://dx.doi.org/10.17654/MS109010201

[9] E.O. Adeyefa, Y. Haruna, R.O. Ajewole and R.I. Ndu: On polynomials construction, International Journal of Mathematical Analysis, 2018, 12(6), 251 – 257.

[10] E.O. Adeyefa, Akintunde, A.O., R.I. Ndu, Oladunjoye, J.A. and Ibrahim, A.A.: Block Nystrom Type Method and Its Block Extension for Fourth Order Initial and Boundary Value Problems, International Journal of Mathematical Analysis, 2018, 12(4), 183 – 198. https://doi.org/10.12988/ijma.2018.711148

[11] F.L. Joseph, R.B. Adeniyi and E.O. Adeyefa: A Collocation Technique for Hybrid Block Methods with a Constructed Orthogonal basis for Second Order Ordinary Differential Equations, Global Journal of Pure and Applied Mathematics. 2018, 14(4), 7–27.

[12] E.O. Adeyefa: Orthogonal-based hybrid block method for solving general second order initial value problems, Italian journal of pure and applied mathematics, 2017, 37, 659-672, http://ijpam.uniud.it/journal/onl_2017-37.htm,

[13] J.D. Lambert, Computational Methods in Ordinary Differential Equation. John Wiley & Sons Inc, 1973.

[14] C. Lanczos, Trigonometric interpolation of empirical and analytical functions. J. Math. Physics, 1938, 17, 123 – 199.

[15] O.R. Folaranmi, R.B. Adeniyi and E.O. Adeyefa: An Orthogonal Based Self-Starting Numerical Integrator for Third Order IVPs in ODEs, The Pacific Journal of Science and Technology, 2016, 17(2), 73-86. http://www.akamaiuniversity.us/PJST17_2_73.pdf.

[16] W.E. Milne, Numerical Solution of Differential Equations. John Wiley and Sons, 1953.

[17] J.B. Rosser, Runge-Kutta for all seasons. SIAM,1967, (9): 417-452.

[18] L.F. Shampine, H.A. Watts, Block Implicit One-Step Methods. Journal of Math of Computation, 1969, 23(108), 731-740. doi:10.1090/S0025-5718-1969-0264854-5.

[19] Y.A. Yahaya, A.M. Badmus, A class of collocation methods for general second order ordinary differential equations. African Journal of Mathematics and Computer Science, 2009, 2(4), 069-072.

[20] M. Zennaro, One-step collocation: Uniform superconvergence, predictor-corrector method, local error estimate, SIAM J. Numerical Analysis, 1985, 22, 1135-1152.