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Three-Step Implicit Block Method for Second Order Odes

[1]E.O. Adeyefa, [2]F.L. Joseph and [1]O.D. Ogwumu
[1]Department Of Mathematics and Statistics, Federal University Wukari, Taraba State, Nigeria.
[2]Department Of Mathematics and Statistics,Bingham University, Karu, Nasarawa State, Nigeria.

ABSTRACT : *This paper focuses on the development of three-step implicit numerical method capable of solving second order Initial Value Problems of Ordinary Differential Equations. The collocation and interpolation techniques are used in the derivation of the scheme and the Chebyshev polynomial is employed as basis function. The scheme is applied as simultaneous integrator to second order initial value problem of ODEs. The self-starting method developed which is capable of producing several outputs of solution at the off-grid points without requiring additional interpolation, was implemented as block method so as to obtain solutions at both step and offstep points. Numerical examples are presented to portray the applicability and the efficiency of the method.*

***KEYWORDS**: Chebyshev polynomial, Collocation, Hybrid, Interpolation, Block Method.*

I. INTRODUCTION

The solutions of second order Initial Value Problems (IVPs) of Ordinary Differential Equations (ODEs) have received much attention by researchers. Many of such problems may not be easily solved analytically, hence numerical schemes are developed to approximate the solution. The approach of reducing (1) to a system of two first order differential equations has been reported to increase the dimension of the problem and therefore result in more computation, [1]. [2], [3],[4], [5] to mention a few have attempted the solution of this kind of problem using LMMs without reduction to system of first order ODEs. Conventionally, implicit LMMs, when implemented in the predictor-corrector mode is prone to error propagation. This disadvantage has led to the development of block methods from linear multistep methods. Apart from being self-starting, the method does not require the development of the predictors separately, and evaluates fewer functions per step. In what now immediately follows, we shall develop a three-step point with Chebyshev polynomial as basis function.

II. PROCEDURE

In this section, we shall consider the derivation of the proposed continuous three-step block method which will be used to generate the main method and other methods required to set up the block method. This we do by approximating the analytical solution of :

$$y'' = f(x, y, y'),\ y'(a) = z_0,\ y(a) = y_0 \qquad (1)$$

where f is a continuous function, with a Chebyshev polynomial in the form:

$$y(x) = \sum_{j=0}^{(r+s)-1} a_j T_j(x) \qquad (2)$$

on the partition $a = x_0 < x_1 < ... < x_n < x_{n+1} < ... < x_N = b$ of the integration interval [a, b], with a constant step size h, given by $h = x_{n+1} - x_n$; $n = 0, 1, ..., N - 1$. The second derivative of (2) is given by:

$$y(x) = \sum_{j=0}^{(r+s)-1} a_j T_j''(x) \qquad (3)$$

where $x \in [a, b]$, the a_j's are real unknown parameters to be determined and r + s is the sum of the number of collocation and interpolation points.

Conventionally, we need to interpolate at at least two points to be able to approximate (2) and, to make this happen, we proceed by arbitrarily selecting some step points, x_{n+v}, $v \in (0, 3)$, in (x_n, x_{n+1}) in such a manner that the zero-stability of the main method is guaranteed. Then (2) is interpolated at x_{n+i}, i = 0, v and its second derivative is collocated at x_{n+i}, i = 0 and 1, so as to obtain a system of equations which will be solved by Gaussian elimination method.

2.1 Derivation of Three-step Points

In this case, three-step points are introduced. Here, i = 2 so that , $v_1 = 0$ and $v_2 = 1$, the collocation point, r = 4 and the interpolation point, s = 2.

From (2), for r = 4 and s = 2, we obtain the polynomial of degree r + s - 1 as:

$$y(x)=\sum_{j=0}^{5} a_j T_j(x) \tag{4}$$

with its second derivative given by:

$$y(x)=\sum_{j=0}^{5} a_j T_j^{"}(x) \tag{5}$$

Substituting (5) into (1) gives:

$$y(x)=\sum_{j=0}^{5} a_j T_j^{"}(x) = f(x,y,y') \tag{6}$$

Collocating (6) at x_{n+0}, x_{n+1}, x_{n+2}, x_{n+3} and interpolating (4) at x_{n+0}, x_{n+1} lead to a system of equations written in matrix form AX = B as follows:

$$\begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 16 & -96 & 320 & -800 \\ 0 & 0 & 16 & 96 & 320 & 800 \\ 0 & 0 & 16 & 288 & 3392 & 33120 \\ 0 & 0 & 16 & 480 & 9536 & 157600 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+1} \\ h^2 f_n \\ h^2 f_{n+1} \\ h^2 f_{n+2} \\ h^2 f_{n+3} \end{pmatrix} \tag{7}$$

Solving (7) by Gaussian elimination method yields the a_j's as follows:

$$\begin{aligned} a_0 &= \frac{1}{2}y_n + \frac{1}{2}y_{n+1} + \frac{h^2}{6144}\{95f_{n+2} - 325f_{n+1} - 135f_n - 19f_{n+3}\} \\ a_1 &= \frac{1}{2}y_{n+1} - \frac{1}{2}y_n + \frac{h^2}{92160}\{469f_n - 447f_{n+1} - 33f_{n+2} + 11f_{n+3}\} \\ a_2 &= \frac{h^2}{1536}\{33f_n + 83f_{n+1} - 25f_{n+2} + 5f_{n+3}\} \\ a_3 &= \frac{h^2}{36864}\{177f_{n+1} - 187f_n + 15f_{n+2} - 5f_{n+3}\} \\ a_4 &= \frac{h^2}{6144}\{3f_n - 7f_{n+1} + 5f_{n+2} - f_{n+3}\} \\ a_5 &= \frac{h^2}{61440}\{3f_{n+1} - f_n - 3f_{n+2} + f_{n+3}\} \end{aligned} \tag{8}$$

Substituting the a_j's, j = 0(**Error! Reference source not found.**)5 into (4) yields the continuous three-step method in the form of a continuous linear multistep method described by the formula:

$$y(x) = \sum_{j=0}^{1} \alpha_j y_{n+j} + h^2 \sum_{j=0}^{3} \beta_j f_{n+j} \tag{9}$$

where α_j's and β_j's are continuous functions and are obtained as parameters:

$$\alpha_0(t) = \frac{1}{2} - \frac{1}{2}t$$

$$\alpha_1(t) = \frac{1}{2} + \frac{1}{2}t$$

$$\beta_0(t) = \frac{-11}{256} + \frac{233}{11520}t + \frac{5}{128}t^2 - \frac{23}{1152}t^3 + \frac{1}{256}t^4 - \frac{1}{3840}t^5 \qquad (10)$$

$$\beta_1(t) = \frac{-83}{768} - \frac{73}{3840}t + \frac{15}{128}t^2 + \frac{7}{384}t^3 - \frac{7}{768}t^4 + \frac{1}{1280}t^5$$

$$\beta_2(t) = \frac{25}{768} - \frac{7}{3840}t - \frac{5}{128}t^2 + \frac{1}{384}t^3 + \frac{5}{768}t^4 - \frac{1}{1280}t^5$$

$$\beta_3(t) = \frac{-5}{768} + \frac{7}{11520}t + \frac{1}{128}t^2 - \frac{1}{1152}t^3 - \frac{1}{768}t^4 + \frac{1}{3840}t^5$$

where t = $\frac{2v-h}{h}$.

Evaluating (9) at x = x_{n+2} and x_{n+3}, we obtain the discrete methods from (9) as:

$$y_{n+3} = 3y_{n+1} - 2y_n + \frac{h^2}{12}(2f_n + 21f_{n+1} + 12f_{n+2} + f_{n+3}) \qquad (11)$$

and:

$$y_{n+2} = 2y_{n+1} - y_n + \frac{h^2}{12}(f_n + 10f_{n+1} + f_{n+2}) \qquad (12)$$

The block methods are derived by evaluating the first derivative of (9) in order to obtain additional equations needed to couple with (11) and (12).

Differentiating (9), we obtain:

$$y'(x) = \sum_{j=0}^{1} \alpha'_j y_{n+j} + h^2 \sum_{j=0}^{3} \beta'_j f_{n+j} \qquad (13)$$

Evaluating (13) at x = x_n, x_{n+1}, x_{n+2} and x_{n+3}, the following discrete derivative schemes are obtained:

$$hy'_n = -y_n + y_{n+1} + \frac{h^2}{360}(-97f_n - 114f_{n+1} + 39f_{n+2} - 8f_{n+3})$$

$$hy'_{n+1} = -y_n + y_{n+1} + \frac{h^2}{360}(38f_n + 171f_{n+1} - 36f_{n+2} + 7f_{n+3}) \qquad (14)$$

$$hy'_{n+2} = -y_n + y_{n+1} + \frac{h^2}{360}(23f_n + 366f_{n+1} + 159f_{n+2} - 8f_{n+3})$$

$$hy'_{n+3} = -y_n + y_{n+1} + \frac{h^2}{360}(38f_n + 57f_{n+1} + 444f_{n+2} + 127f_{n+3})$$

Equations (11), (12) and (14) are combined and solved simultaneously to obtain the following explicit results:

$$y_{n+1} = hy'_n + y_n - \frac{h^2}{360}\{-97f_n - 114f_{n+1} + 39f_{n+2} - 8f_{n+3}\}$$

$$y_{n+2} = 2hy'_n + y_n + \frac{h^2}{45}\{28f_n + 66f_{n+1} - 6f_{n+2} + 2f_{n+3}\}$$

$$y_{n+3} = 3hy'_n + y_n + \frac{h^2}{40}\{39f_n + 108f_{n+1} + 27f_{n+2} + 6f_{n+3}\}$$

$$y'_{n+1} = y'_n + \frac{h}{24}\{9f_n + 19f_{n+1} - 5f_{n+2} + f_{n+3}\}$$

$$y'_{n+2} = y'_n + \frac{h}{3}\{f_n + 4f_{n+1} + f_{n+2}\}$$

$$y'_{n+3} = y'_n + \frac{h}{360}\{135f_n + 171f_{n+1} + 405f_{n+2} + 135f_{n+3}\}$$

III. THE BASIC PROPERTIES OF THE METHOD

3.1 Order and Error Constant

The explicit schemes (11) and (12) derived are discrete schemes belonging to the class of LMM of the form:

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h^2\sum_{j=0}^{k}\beta_j f_{n+j} \qquad (15)$$

Associated with (12) is the linear differential operator L defined by:

$$L[y(x);h] = \sum_{j=0}^{k}[\alpha_j y(x+jh) - h^2\beta_j y''(x+jh)] \qquad (16)$$

Expanding (13) by Taylor series, we have:

$$L[y(x);h] = C_0 y(x) + C_1 h y'(x) + \ldots + C_q h^q y^{(q)}(x) + \ldots \qquad (17)$$

where:

$$C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \ldots + \alpha_k$$

$$C_1 = \alpha_1 + 2\alpha_2 + \ldots + k\alpha_k$$

$$C_2 = \frac{1}{2!}(\alpha_1 + 2^2\alpha_2 + \ldots + k^2\alpha_k) - (\beta_0 + \beta_1 + \beta_2 + \ldots + \beta_k)$$

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$$C_p = \frac{1}{p!}(\alpha_1 + 2^p\alpha_2 + \ldots + k^p\alpha_k) - \frac{1}{(q-2)!}(\beta_1 + 2^{q-2}\beta_2 + \ldots + k^{q-2}\beta_k), q \geq 3$$

The LMM (15) is said to be of order p if $C_0 = C_1 = C_2 = \ldots = C_p = C_{p+1} = 0$ and $C_{p+2} \neq 0$ is the error constant, see Lambert (1973). According to this definition, the discrete schemes (11) and (12) have order $p = (4, 4)^T$ with error constants, $C_{p+2} = (\frac{1}{80}, \frac{1}{240})^T$.

3.2 Consistency of the Method

The LMM (15) is said to be consistent if it is of order $p \geq 1$ and its first and second characteristic polynomials defined as $\rho(z) = \sum_{j=0}^{k}\alpha_j z^j \; and \; \sigma(z) = \sum_{j=0}^{k}\beta_j z^j$ where z satisfies:

$$(i)\sum_{j=0}^{k}\alpha_j = 0, (ii)\,\rho(1) = \rho'(1) = 0, (iii)\,\rho''(1) = 2!\sigma(1), see\, Lambert\,(1973).$$

Investigation reveals that the schemes (11) and (12) satisfied these conditions.

3.3 Zero-Stability of the Method

The LMM (12) is said to be zero-stable if no root of the first characteristic polynomial has modulus greater than one, and if every root of modulus one has multiplicity not greater than two, see Lambert (1973).

The schemes (11) and (12) have been investigated to satisfy this definition.

4. Numerical Examples

Two test problems are considered here to examine the efficiency and accuracy of the method implemented as a block method. The absolute errors of the test problems are compared with existing methods.

4.1 Problems

Problem 1

$$y'' = y + xe^{3x},\; y(0) = \frac{-3}{32},\; y'(0) = \frac{-5}{32},\; h = 0.0025$$

$$Exact\ Solution: y(x) = \frac{4x-3}{32e^{-3x}}$$

$$Source: Adesanya\ et\ al\,(2009)$$

Probem 2

$$y'' = 2y - y', \; y(0) = 0, \; y'(0) = 1$$

$$Exact\ Solution: y(x) = \frac{e^{x} - e^{-2x}}{3}$$

$$Source: Adeniyi\ et\ al\ (2008).$$

4.2 Results

Table 1: Showing the Exact Solutions and Absolute Error for Problem 1

X	Exact Solution	Three-Step Points	Error	Error in [4]
0.0025	-0.094140915761849	-0.094140915761848	4.163336342344337e-016	7.020D-14
0.0050	-0.094532404142339	-0.094532404142338	9.436895709313831e-016	1.217D-13
0.0075	-0.094924451608388	-0.094924451608386	1.582067810090848e-015	3.396D-12
0.0100	-0.095317044390700	-0.095317044390696	-4.468647674116255e-015	8.122D-12
0.0125	-0.095710168480981	-0.095710168480977	3.635980405647388e-015	1.453D-11

Table 2: Showing the Exact Solutions and Absolute Error for Problem 2

X	Exact	Three steps	Error	Error in [7]
0.1	0.095470288185115	0.095471666666667	9.766814106484945e-006	3.7135900e-4
0.2	0.183675922345822	0.183804444444444	1.831502902138738e-005	1.5836410e-3
0.3	0.266950622579730	0.266726074074074	6.510124759606661e-005	3.4173760e-3
0.4	0.347396099439354	0.386643055440329	1.024784019953984e-004	5.7170630e-3

IV. CONCLUSION

In this paper, the derivation of continuous three-step method for numerical solution of second order IVPs of ODEs without reformulation to first order system has been considered. In TABLE 1, we compare the block method of Adesanya et al while in TABLE 2, the predictor-corrector method of Adeniyi et al is compared. The results in both cases show that our new method is more efficient on comparison. Moreover, the desirable property of a numerical solution is to behave like the theoretical solution of the problem as this is obvious in TABLES 1 and 2. In the future paper, the scope of the paper shall be extended to hybrid three-step points.

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