A CONTINOUS COLLOCATION FORMULATION OF TWO-STEP IMPLICIT BLOCK METHOD FOR SECOND ORDER ODEs USING LEGENDRE BASIS FUNCTION

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ABSTRACT
The numerical solution of Initial Value Problem of general second order Ordinary Differential Equations have been studied in this work. Implicit two-step continuous multistep method for solving this type of problem has been developed by collocation and interpolation technique. The continuous method which employs Legendre polynomial as basis function yields discrete equivalent as a block method. The main scheme obtained was implemented together with the block formulae for the numerical solution of second order differential equations. Numerical examples are presented to illustrate the applicability and efficiency of the method. The results obtained, when compared with existing methods, are favourable.

Keywords: *Implicit block method, Legendre basis function, Collocation, Interpolation, Second order ODE*
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INTRODUCTION

Differential equations arise in science, engineering, and diverse fields such as medicine, economics, operations research, etc. These are mathematical models that are developed to help in the understanding of physical phenomena. This work is concerned with the study of numerical solutions of initial value problems in second-order ordinary differential equations. The Initial Value Problems (IVP) in ordinary differential equations (ODE) are of the form:

$$y'' = f(x, y, y'), \qquad y(a) = y_0, \qquad y'(a) = z_0, \qquad x \in [a, b] \tag{1}$$

Some of these problems may not be easily solved analytically; hence, numerical schemes are developed to approximate the solution. The method of reducing to a system of first-order differential equations has been reported to increase the dimension of the problem and therefore result in more computational (see Bun and Varsilyer, 1992).

Many researchers have used block methods. Anake et al. (2012) used power series as the basis function. Folaranmi et al. (2017) used Chebyshev as the basis function. Also, Adeniyi et al. (2008), Awoyemi and Kayode (2008), Sunday et al (2022), to mention but a few, have all worked on the collocation method for solving equations.

The implicit LMMs, when implemented in the predictor-corrector mode, are prone to error propagation. This disadvantage led to the development of block methods from linear multi-step methods. Apart from being self-starting, the method does not require the development of the predictors separately and evaluates fewer functions per step.

The present step is an extension of the works of Anake et al. (2012) and Folaranmi et al. (2017) about the development of hybrid block methods for IVP. A Legendre polynomial basis will be employed for the formation of block hybrid schemes.

DEVELOPMENT OF THE METHOD

For this purpose, we shall approximate the analytical solution of (1) with an approximant of the form

$$y(x) = \sum_{j=0}^{r+s-1} a_j P_j\,(x) \tag{2}$$

Where $a_j's$ are constants to be determined, r is the number of collocation points and s is the number of interpolation points, on the partition

$$a = x_0 < x_1 < \cdots < x_n < x_{n+1} < \cdots < x_n = b$$

Of the integration interval $[a, b]$ with a constant step size h, given by

$$h = x_{n+1} - x_n : n = 0,1, \dots N - 1$$

The function $P_j(x)$ is the $j - th$ degree Legendre polynomial value in the range of integration of (1), that is, in $[a, b]$.

The second derivative of (2) is given by

$$y''(x) = \sum_{j=0}^{r+s-1} a_j\, P''_j(x) \tag{3}$$

Where $x \in [a, b]$ and $r + s$ is the sum of the number of collocation and interpolation points. We shall interpolate at least two points to be able to approximate (1) and, to make this happen, we proceed by selecting some points x_{n+v}, where $v \in (0, n)$. Then (2) is interpolated at x_{n+s} and its second derivative is collocated at x_{n+r}, so as to obtain a system of equations which will be solved by Gaussian elimination method. We shall consider this method for non-hybrid and hybrid methods.

2.1 DEVELOPMENT OF TWO-STEP METHOD

In deriving this method, we set $s = 2$ and $r = 3$ in (2) and (3) so as to obtain a system of five equations, each of degree four as follows:

$$\sum_{j=0}^{4} a_j P_j\,(x) = y(x) \tag{4}$$

$$\sum_{j=0}^{4} a_j\, P''_j(x) = f(x, y, y') \tag{5}$$

We now collocate (5) at $x = x_{n+s} = 0,1\ and\ 2$, and interpolate (4) at $x = x_{n+r}, r = 0,1\ and\ 2$ to have a system of equations written in the matrix form $AX = B$ as,

$$\begin{bmatrix} 1 & -3 & 13 & -63 & 321 \\ 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 12 & -180 & 1860 \\ 0 & 0 & 12 & -60 & 180 \\ 0 & 0 & 12 & 60 & 180 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+1} \\ h^2 f_n \\ h^2 f_{n+1} \\ h^2 f_{n+2} \end{bmatrix} \tag{6}$$

The equation (6) above is solved by Guassian elimination method to obtain the value of the unknown parameters $a_j, j = 0,1, \dots 4$ as follows:

$$\left.\begin{aligned} a_0 &= \frac{3}{2} y_{n+1} - \frac{1}{2} y_n + \frac{h^2}{20} f_n + \frac{43h^2}{120} f_{n+1} + \frac{h^2}{120} f_{n+2} \\ a_1 &= \frac{1}{2} y_{n+1} - \frac{1}{2} y_n + \frac{h^2}{24} f_n + \frac{17h^2}{40} f_{n+1} + \frac{h^2}{30} f_{n+2} \\ a_2 &= \frac{5h^2}{88} f_{n+1} - \frac{h^2}{112} f_n + \frac{11h^2}{336} f_{n+2} \\ a_3 &= \frac{h^2}{120} f_{n+2} - \frac{h^2}{120} f_{n+1} \\ a_4 &= \frac{h^2}{1680} f_n - \frac{h^2}{840} f_{n+1} + \frac{h^2}{1680} f_{n+2} \end{aligned}\right\} \tag{7}$$

Substituting (7) into (2) yields a continuous implicit two step method in the form

$$y(x) = \alpha_0(x) y_n + \alpha_1(x) y_{n+1} + h^2 \sum_{j=0}^{2} \beta_j(x) f_{n+j} \tag{8}$$

where $\alpha_j(x)$ and $\beta_j(x)$ are continuous coefficients. From (8), we get the parameters $\alpha_j(x)$ and $\beta_j(x)$ as:

$$
\left.\begin{aligned}
\alpha_0 &= \frac{(2h-2x+2x_n)}{2h} \\
\alpha_1 &= \frac{(1-(2h-2x+2x_n)}{2h} \\
\beta_0 &= \frac{(h-x+x_n)^3}{12h^3}+\frac{(h-x+x_n)^4}{24h^4}-\frac{(2h-2x+2x_n)}{24h} \\
&\quad -\frac{(6h-6x+6x_n)}{112h}+\frac{(20h-20x+20x_n)}{1680h} \\
\beta_1 &= \frac{(h-x+x_n)^2}{2h^2}-\frac{(h-x+x_n)^4}{12h^4}-\frac{\big(17(2h-2x+2x_n)\big)}{40h} \\
&\quad +\frac{(5(6h-6x+6x_n))}{84h}+-\frac{(12h-12x+12x_n)}{120h}-\frac{(20h-20x+20x_n)}{840h} \\
\beta_2 &= \frac{(h-x+x_n)^4}{24h^4}-\frac{(h-x+x_n)^4}{12h^3}-\frac{(2h-2x+2x_n)}{30h} \\
&\quad +\frac{(11(6h-6x+6x_n))}{336h}-\frac{(12h-12x+12x_n)}{120h}+\frac{(20h-20x+20x_n)}{1680h}
\end{aligned}\right\} \qquad (9)
$$

Evaluating (8) at x_{n+2}, the main method is obtained as

$$
y_{n+2}+y_n-2y_{n+1}=\frac{h^2}{12}(f_n+10f_{n+1}+f_{n+2}) \qquad (10)
$$

The block methods are derived by evaluating the first derivative of (8) in order to obtain additional equations needed to couple with (10).

Differentiating (8) we obtain

$$
y'(x)=\sum_{j=0}^{1}\alpha_j{}'(x)y_{n+j}+\sum_{j=0}^{2}\beta_j{}'(x)f_{n+j} \qquad (11)
$$

Where

$$\left.\begin{array}{c}\alpha'_0=\dfrac{-1}{h}\\ \alpha_1=\dfrac{1}{h}\\ \beta'_0=\dfrac{1}{18h}-\dfrac{(h-x+x_n)^3}{6h^4}-\dfrac{(h-x+x_n)^2}{4h^3}+\dfrac{(2h-12x+12x_n)}{112h^2}-\dfrac{(180h-180x+180x_n)}{1680h^2}\\ \beta'_1=\dfrac{(h-x+x_n)^3}{3h^4}+\dfrac{5}{12h}-\dfrac{\left(5(12h-14x+12x_n)\right)}{84h^2}-\dfrac{(60h-60x+60x_n)}{120h^2}+\dfrac{(180h-180x+80x_n)}{120h^2}\\ \beta'_2=\dfrac{(h-x+x_n)^2}{4h^3}-\dfrac{(h-x+x_n)^3}{6h^4}-\dfrac{1}{24h}-\dfrac{(11(2h-12x+3x_n))}{336h^2}\end{array}\right\} \quad (12)$$

Evaluating (12) at x_n, x_{n+1} and x_{n+2}, we get the following discrete derivative schemes:

$$hy'_n=-y_n+y_{n+1}-\frac{h^2}{24}(7f_n+6f_{n+1}-f_{n+2}) \quad (13)$$

$$hy'_{n+1}=-y_n+y_{n+1}+\frac{h^2}{24}(3f_n+10f_{n+1}-f_{n+2}) \quad (14)$$

$$hy'_{n+2}=-y_n+y_{n+1}+\frac{h^2}{24}(f_n+20f_{n+1}+9f_{n+2}) \quad (15)$$

Solving equations (10), (13), (14) and (15) simultaneously, we obtain the following explicit results

$$y_{n+1}=y_n+hy'_n+\frac{h^2}{24}(7f_n+6f_{n+1}-f_{n+2}) \quad (16)$$

$$y_{n+2}=y_n+2hy'_n+\frac{2h^2}{3}(f_n+2f_{n+1}) \quad (17)$$

$$y'_{n+1}=y'_n+\frac{h}{12}(5f_n+8f_{n+1}-f_{n+2}) \quad (18)$$

$$y'_{n+2}=y'_n+\frac{h}{3}(f_n+4f_{n+1}+f_{n+2}) \quad (19)$$

3. ANALYSIS OF THE METHOD

In analyzing the properties of this method, we consider (10) and (16) − (19) in order to determine the order, error constant, consistency and zero stability of the two-step method. Equation (10) derived is a discrete scheme belonging to the class of (20) $LMMs$ of the form

$$\sum_{j=0}^{k}\alpha_j\,y_{n+j}=h^2\sum_{j=0}^{k}\beta_j\,f_{n+j} \quad (20)$$

3.1 Order and Error Constant

Consider the $LMM(20)$ associated with the linear difference operator L defined by

$$L[y(x);h]=\sum_{j=0}^{k}[\alpha_j y\,(x+jh)-h^2\beta_j y''(x+jh)] \qquad (21)$$

Where $y(x)$ is an arbitrary function, continuously differentiable on $[a,b]$.
Expanding (21) by Taylor series, we have

$$L[y(x);h]=C_0y(x)+C_1y'(x)+\cdots+C_qh^qy^{(q)}(x)+\cdots \qquad (22)$$

where

$$C_0=\alpha_0+\alpha_1+\alpha_2+\cdots+\alpha_k$$
$$C_1=\alpha_1+2\alpha_2+\cdots+k\alpha_k$$
$$C_2=\frac{1}{2!}(\alpha_1+2^2\alpha_2+\cdots+k^2\alpha_k)-(\beta_0+\beta_1+\beta_2+\cdots+\beta_k)$$

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$$C_q=\frac{1}{q!}(\alpha_1+2^q\alpha_2+\cdots+k^q\alpha_k)-\frac{1}{(q-2)}(\beta_1+2^{q-2}\beta_2+\cdots+k^{q-2}\beta_k), \qquad q\ge 3$$

The method is of **order $\boldsymbol{p}$** if
$C_0=C_1=C_2+\cdots+=C_p=C_{p+1}=0$ and $C_{p+2}\neq 0$.
The $C_{p+2}\neq 0$ is called the **error constant**, and $C_{p+2}h^{p+2}y^{(p+2)}(x_n)$ is the **principal local truncation error** at the point x_n.
We express (10) in the form

$$y_{n+2}+y_n-2y_{n+1}=\frac{h^2}{12}(f_n+10f_{n+1}+f_{n+2})=0 \qquad (23)$$

Expressing (23) in Taylor series, we have

$$y_{n+2}=y_n+2hy'+\frac{(2h)^2y''}{2!}+\frac{(2h)^3y'''}{3!}+\frac{(2h)^4y^{iv}}{4!}+\frac{(2h)^5y^{v}}{5!}+\frac{(2h)^6y^{vi}}{6!}+\frac{(2h)^2y^{vii}}{7!} \qquad (24)$$

Which gives

$$C_0=1-1=0$$
$$C_1=1-1=0$$
$$C_2=\frac{3}{2}-\frac{3}{2}=0$$
$$C_3=\frac{4}{2}-\frac{2}{3}-\frac{4}{3}=0$$
$$C_3=\frac{4}{2}-\frac{2}{3}-\frac{4}{3}=0$$
$$C_4=\frac{8}{6}-\frac{4}{3}=0$$

$$C_5 = \frac{16}{24} - \frac{4}{6} = 0$$
$$C_6 = \frac{32}{120} - \frac{4}{18} = \frac{2}{45}$$

Thus, equation (10) is of order $p = 4$ and error constant $C_{p+2} = \frac{2}{45}$

3.2 Consistency

The linear multistep method (20) is said to be consistent if it has order $p \geq 1$ and the first and second characteristic polynomials which are defined as

$$\rho(z) = \sum_{j=0}^{k} \alpha_j z^j$$
$$\sigma(z) = \sum_{j=0}^{k} \beta_j z^j$$

where z, the principal root, satisfies the following conditions:

$$(i) \sum_{j=0}^{k} \alpha_j = 0$$
$$(ii)\ \ \rho(1) = \rho'(1) = 0$$

and

$$\rho''(1) = 2!\,\sigma(1)$$

The two-step method derived is of order $p = 4 > 1$.
Conditions $(i) - (iii)$ are investigated below.
In equation (10),

$$\alpha_2 = 1, \alpha_1 = -2 \; and \; \alpha_0 = 1$$
$$\sum \alpha_j = 1 - 2 + 1 = 0, \qquad j = 0,1,2$$
$$\rho(z) = z^2 - 2z + 1$$
$$\sigma(z) = \frac{z^2 - 2z + 1}{12}$$
$$\rho(1) = 1 - 2 + 1 = 0$$
$$\rho'(z) = 2z - 2$$
$$\rho'(1) = 2(1) - 2 = 0$$

Therefore, $\rho(1) = \rho'(1) = 0$.
Also, $\rho''(z) = 2$
Hence, $\rho''(1) = 2$

$$\sigma(1) = \frac{1 + 10 + 1}{12} = 1$$
$$2!\,\sigma(1) = 2$$

Therefore, $\rho''(z) = 2!\,\sigma(1) = 2$
Since the conditions are satisfied then, the method is consistent.

3.3 Zero-Stability

The linear multistep method (20) is said to be zero-stable if no root of the first characteristic polynomial $\rho(z)$ has modulus greater than one, and if every root of modulus one has multiplicity not greater than two, see (Lambert, 1973).
The first characteristic polynomial of the method is defined as

$$\rho(z) = z^2 - 2z + 1$$

Equating $\rho(z)$ to zero and solving for z, we have $z = 1\ (twice)$.
Hence, $|z| = 1$ is simple. The method is zero stable.
The roots of the derived block method have been verified to be less than or equal to 1 and $|z| = 1$. Therefore, the scheme is zero stable.

4. Numerical Examples

We consider here the application of the derived scheme on four tests problems.

Problem 1

$$y'' - 100y = 0, 0 \leq x \leq 0.12$$
$$y(0) = 1, y'(0) = -10$$
$$h = 0.01$$
$$Exact\ Solution: y = e^{-10x}$$
$$\boldsymbol{Source}: \boldsymbol{Areo}\ (\mathbf{2013})$$

Problem 2

$$y'' = 2y - y', 0 \leq x \leq 0.4$$
$$y(0) = 0, y'(0) = 1$$
$$h = 0.1$$
$$Closed\ form\ solution: y(x) = \frac{e^x - e^{-2x}}{3}$$
$$\boldsymbol{Source}: \boldsymbol{Adeniyi\ et\ al}\ (\mathbf{2008})$$

Problem 3 (A Nonlinear Problem)

$$y'' - (y')^2 = 0, 0 \leq x \leq 0.03125$$
$$y(0) = 1, y'(0) = \frac{1}{2}$$
$$h = 0.003125$$
$$Analytical\ Solution: y(x) = 1 + \frac{1}{2} ln \frac{2 + x}{2 - x}$$

Problem 4

$$y'' + y = 0, 0 \leq x \leq 1.2$$
$$y(0) = 1, y'(0) = 1$$

$$h = 0.1$$
$$True\ Solution: y(x) = cosx + sinx$$
$$\boldsymbol{Source: Yahaya\ and\ Badmus\ (2009)}$$

4.1 Tables of Results

Table 1: Numerical Results for Problem 1

$\boldsymbol{X}$	$\boldsymbol{New\ Method}$	$\boldsymbol{Exact\ Solution}$	$\boldsymbol{Error}$
0.01	0.9048376227	0.9048374180	$2.0466404049e^{-007}$
0.02	0.8187311683	0.8187307531	$4.1522201821e^{-007}$
0.03	0.7408188198	0.7408182207	$5.9911828210e^{-007}$
0.04	0.6703208381	0.6703200460	$7.92064360656e^{-007}$
0.05	0.6065316274	0.6065306597	$9.6768736657e^{-007}$
0.06	0.5488127917	0.5488116361	$1.1556059734e^{-006}$
0.07	0.4965866382	0.4965853038	$1.3344085905e^{-006}$
0.08	0.4493304927	0.4493289641	$1.5285827784e^{-006}$
0.09	0.3328732333	0.4065696597	$1.7211594008e^{-006}$
0.10	0.3011966014	0.3678794412	$1.9326285576e^{-006}$
0.11	0.2725344350	0.3328710837	$-2.1496019205e^{-006}$
0.12	0.2465998860	0.3011942119	$-2.389487797904e^{-006}$

Table 2: Numerical Results for Problem 2

X	$\boldsymbol{New\ Method}$	$\boldsymbol{Exact\ Solution}$	$\boldsymbol{Error}$

0.1	0.09547805317	0.095480055	$2.0018292220e^{-006}$
0.2	0.1836904268	0.1836942374	$3.8105748434e^{-006}$
0.3	0.2670106955	0.2670157238	$5.0283273255e^{-006}$
0.4	0.3474923823	0.3474985778	$6.1955413495e^{-006}$

Table 3: Numerical Results for Problem 3

X	$New\ Method$	$Exact\ Solution$	$Error$
0.003125	1.001563722	1.0015625012	$1.2207284330e^{-006}$
0.006250	1.003129893	1.0031250102	$4.8828274141e^{-006}$
0.009375	1.004698521	1.0046875343	$1.0986667271e^{-005}$
0.0125	1.006269613	1.0062500814	$1.9531617883e^{-005}$
0.015625	1.007843177	1.0078126590	$3.0518048460e^{-005}$
0.01875	1.009419222	1.0093752747	$4.3947327311e^{-005}$
0.021875	1.010997754	1.0109379362	$5.9817821637e^{-005}$
0.025	1.012578782	1.0125006511	$7.8130897291e^{-005}$
0.028125	1.014162314	1.0140634271	$9.8886918562e^{-005}$
0.03125	1.015748357	1.0148564487	$8.9190834074e^{-004}$

Table 4: Numerical Results for Problem 4

X	$New\ Method$	$Exact\ Solution$	$Error$

0.1	1.094837380	1.0948375819	$2.0192485274e^{-007}$
0.2	1.178735502	1.1787359086	$4.0663630285e^{-007}$
0.3	1.250856132	1.2508566958	$5.6378694601e^{-007}$
0.4	1.310478617	1.3104793363	$7.1931153589e^{-007}$
0.5	1.357007281	1.3570081005	$8.1949457597e^{-007}$
0.6	1.389977171	1.3899780883	$9.1730471396e^{-007}$
0.7	1.409058921	1.4090598745	$9.5352218010e^{-007}$
0.8	1.414061814	1.4140628002	$9.8624668787e^{-007}$
0.9	1.404935922	1.4049368779	$9.5589814796e^{-007}$
1.0	1.381772368	1.3817732907	$9.2267603601e^{-007}$
1.1	1.344802653	1.3448034815	$8.2848701299e^{-007}$
1.2	1.294396109	1.2943968404	$7.3144389989e^{-007}$

5. Conclusion

The continuous non-hybrid two-step method has been developed by the interpolation and collocation technique with Legendre as the basis function. The analysis of the method has also been carried out, and the derived schemes are zero-stable and consistent; hence they are convergent. Four test problems have been considered to test the efficiency and accuracy of our method. It is obvious from our tables of results that the method is efficient and accurate since the approximation closely approximates the analytic solution.

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