# OPTIMAL CONTROL OF LASSA FEVER QUARANTINE MODEL

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## Abstract

In this paper, Optimal control is applied to a Quarantine Lassa fever model developed. The controls $w_1$, $w_2$ and $w_3$ which represent Public education on lassa fever mode of transfer, use of Standard precautions in treatment and Environmental sanitation respectively. The existence and necessary condition for Optimal control of the disease were verified. The effects of these controls on the model are dipitched in graphical representation showing with or without controls implementation.

**Keywords:** *Hamiltonian; Pontryagin's maximum principle; Optimality conditions; Quarantine; adjoint*

## 1. Introduction

The virus was discovered in the year 1969 at advent three nurses from America who became infected in a small village Lassa, Borno State of Nigeria (Macher and Wolfe, 2007)[6]. Lassa fever symptoms are manifested within the first twenty-one days after the infection with severe sickness including important organs such as liver and other symptoms are swelling the face, muscle tiredness, vomiting, coughing, meningitis, and hypertension. Neurological issues such as loss of hearing, this can be momentary or perpetual, vibrations, and inflammation of the brain was explained by Omalibu, et al. (2005) [10].

Literature on Optimal Control Measures available for Lassa fever are few. Although some researchers have worked on Optimal controls models of vector-host-parasites and brought up fascinating models along side with various theories of the dynamics of diseases spread and have employed the prevention strategist known as control therapeutic which could be used to prevent diseases and treatment strategies use for treating diseases at a lowest price (Lashari *et al.*, 2012[4]; Ozair*et al.*, 2012[11]; Lashari *et al.*, 2013)[5].

In this paper, optimal control measures of intervention strategies of Lassa ferver such as control measures through public education on the mode of Lassa fever spread, control measures by adopting proper standard precautions in treatment of Lassa fever virus and control measures by proper hygiene and environmental sanitation are employed.

## 2. Lassa fever Quarantine Model Formulation

Mohammed *et al.*, (2014)[8] formulated a Lassa fever comprises human population and reservoir population. A new model is proposed known as the Quarantined Lassa fever model. The total human population is

$$N_h(t) = S_h + I_h + Q_h + R_h \qquad (1)$$

which is made up Susceptible population $S_h$, Infected population $I_h$, Quarantined population $Q_h$ and Recovered human population $R_h$.

The total reservoir population is

$$N_m(t) = S_m + I_m \qquad (2)$$

made up of Susceptible reservoir population $S_m$ and the Infected reservoir population $I_m$.

Incorporating on the Quarantine model, time-dependent preventive measures $w_3, w_2$ and $w_1$ as controls to fight the spread of Lassa fever, where the parameters are environmental sanitation and hygiene, efforts of treatment using standard precautions and public education of the people of the mode of spread respectively, the model becomes:

$$\left.\begin{aligned}
&\frac{dS_m}{dt} = \Lambda - (1-w_3)\beta_3 S_m I_m - \mu_m S_m \\
&\frac{dI_m}{dt} = (1-w_3)\beta_3 S_m I_m + \theta - \mu_m I_m \\
&\frac{dS_h}{dt} = A_h - (1-w_1)(\beta_1 I_m S_h + \beta_2 I_h S_h) + \varepsilon R_h - \mu_h S_h \\
&\frac{dI_h}{dt} = (\beta_1 I_m S_h + \beta_2 I_h S_h)(1-w_1) - \phi I_h - \mu_h I_h - \delta_1 I_h \\
&-(1-w_2)\alpha I_h \\
&\frac{dQ_h}{dt} = (1-w_2)\alpha I_h - \gamma Q_h - (\delta_2 + \mu_h) Q_h \\
&\frac{dR_h}{dt} = \gamma Q_h - \varepsilon R_h - \mu_h R_h + \phi I_h
\end{aligned}\right\} \qquad (3)$$

## 3. Model Analysis

Let an objective function be defined as:

$$J(w_1, w_2, w_3) = \int_0^{t_t} (A_1 S_h + A_2 I_h + A_3 I_m + \frac{1}{2}(a_1 w_1^2 + a_2 w_2^2 + a_3 w_3^2))dt \tag{4}$$

where $A_1, A_2,$ and $A_3$ denote the weight constant of the susceptible humans, infected human and infected mastomys. In addition, the terms $a_1 w_1^2$, $a_2 w_2^2$ and $a_3 w_3^2$ stand for the associated cost of minimization of number of Susceptible, number of the infected and the number of mastomys population. The preference of quadratic cost on the controls is is owned to Agusto *et al.*, (2012)[2] and as used by other literature. The goal is to seek an optimal control $w_1^*, w_2^*$ and $w_3^*$ such that

$$J(w_1^*, w_2^*, w_3^*) = min\{J(w_1^*, w_2^*, w_3^* \mid w_1^*, w_2^*, w_3^* \in W)\} \tag{5}$$

given that $\{w_1^*, w_2^*, w_3^* \mid w_i(t)$ is Lebesgue measurable with with $0 \le w_i(t) \le 1, i = 1,2,3\}$

Applying the Pontryagin's Maximum Principle (PMP) the system (3) and (4) becomes a minimization problem with respect to $w_1^*, w_2^*$ and $w_3^*$ and the Hamiltonian H, where

$$\begin{aligned}
&\left[H(S_m, I_m, S_h, I_h, Q_h, R_h, W, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, t)\right] = A_1 S_h + A_2 I_h + A_3 I_m + \\
&\frac{1}{2}(a_1 w_1^2 + a_2 w_2^2 + a_3 w_3^2) + \lambda_1[\Lambda - (1-w_3)\beta_3 I_m S_m - \mu_m S_m] + \lambda_2((1-w_3)\beta_3 I_m S_m - \mu_m I_m + \theta) \\
&+ \lambda_3[AS_h - (1-w_1)(\beta_1 I_m S_h + \beta_2 I_h S_h) + \varepsilon R_h - \mu_h S_h] + \\
&\lambda_4((\beta_1 I_m S_h + \beta_2 I_h S_h)(1-w_1) - \phi I_h - (\mu_h + \delta_1) I_h - (1-w_2)\alpha I_h) + \\
&\lambda_5((1-w_2)\alpha I_h - \gamma Q_h - (\delta_2 + \mu_h) Q_h) + \lambda_6(\gamma Q_h - \varepsilon R_n - \mu_h R_h + \phi I_h)
\end{aligned} \tag{6}$$

Where $\lambda_i$, for $i = 1...,6$ are adjoints/costate variables obtained from partial differentiation of the Hamiltonian with respect to the corresponding state variables.

### The Existence and the Necessary Condition of Optimal Control of Quarantine Model

*Theorem* 1

Consider the optimal contol problem of (4) with subject to (3) . There exist an Optimal control set $W^* = (w_1^*, w_2^*, w_3^*)$ with initial condition at $t = 0$ such that

$$J(w_1^*, w_2^*, w_3^*) = min J(w_1, w_2, w_3) \mid w_i \in \text{W for I} = 1\ldots,3 \tag{7}$$

*Proof* :

Owing to the fact that all State variables with control parameters are non-negative, the necessary convexity of the objective functional in $w_1^*, w_2^*$ and $w_3^*$ are fulfilled in this minimization problems. The set of all control variables $(w_1^*, w_2^*, w_3^*) \in W$ is also convex and closed by definition. The Optimal control is bounded which determines compactness needed for the existence of the Optimal. In addition, the integral in the functional (4)

$A_1 S_h + A_2 I_h + A_3 I_m + \frac{1}{2}(a_1 w_1^2 + a_2 w_2^2 + a_3 w_3^2)$ is convex on the control set Kahuru *et al.* (2017)[3].

**Necessary condition for Optimality of the Quarantine Model**

*Theorem* 2

Given an Optimal control $w_1^*, w_2^*$ and $w_3^*$ and solutions $S_m^*, I_m^*, S_h^*, I_h^*, Q_h^*,$ and $R_h^*$ of the corresponding state system (3) that minimizes $(w_1^*\ w_2^*\ w_3^*)$ over W. Then there exists adjoint variables $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ satisfying

$$\left.\begin{aligned}
\frac{d\lambda_1}{dt} &= (\lambda_1 - \lambda_2)((1-w_3)\beta_3 I_m) + \lambda_1 \mu_m \\
\frac{d\lambda_2}{dt} &= -A_3 + (\lambda_1 - \lambda_2)((1-w_3)\beta_3 S_m) + \lambda_2 \mu_m + (\lambda_3 - \lambda_4)((1-w_1)\beta_1 S_h) \\
\frac{d\lambda_3}{dt} &= -A_1 - \lambda_3 A + (\lambda_3 - \lambda_4)(\beta_1 I_m + \beta_2 I_h)(1-w_1)) + \lambda_3 \mu_h \\
\frac{d\lambda_4}{dt} &= -A_2 + (\lambda_3 - \lambda_4)(\beta_2 S_h (1-w_2)) + \lambda_4(\phi + \mu_h + \delta_1) + (\lambda_4 - \lambda_5)(1-w_2)\alpha - \lambda_6 \phi \\
\frac{d\lambda_5}{dt} &= \gamma(\lambda_5 - \lambda_6) + \lambda_5(\delta_2 + \mu_h) \\
\frac{d\lambda_6}{dt} &= \varepsilon(\lambda_6 - \lambda_3) + \lambda_6 \mu_h
\end{aligned}\right\} \tag{8}$$

With the tranversality conditions

$$\lambda_1(t_f) = \lambda_2(t_f) = \lambda_3(t_f) = \lambda_4(t_f) = \lambda_5(t_f) = \lambda_6(t_f) = 0$$

and the controls $w_1^*, w_2^*$ and $w_3^*$ satisfying the Optimality conditions:

$$w_1^* = Max[0, min(1, \frac{(\lambda_4 - \lambda_3)(\beta_1^* I_m^* S_h^* + \beta_2^* I_h^* S_h^*)}{a_1})] \tag{9}$$

$$w_2^* = Max[0, min(1, \frac{\alpha I_h^* (\lambda_5 - \lambda_4)}{a_2})] \tag{10}$$

$$w_3^* = Max[0, min(1, \frac{(\lambda_2 - \lambda_1)\beta_3 S_m^* I_m^*}{a_3})] \tag{11}$$

*Proof* :

Differentiating the Hamiltonian function partially, results to differential equations governing the adjoint variable; hence the adjoint system can be written as

$$\left.\begin{aligned}
\frac{-d\lambda_1}{dt} &= \frac{\partial H}{\partial S_m} = (\lambda_1 - \lambda_2)((1 - w_3)\beta_3 I_m) + \lambda_1 \mu_m \\
\frac{-d\lambda_2}{dt} &= \frac{\partial H}{\partial I_m} = -A_3 + (\lambda_1 - \lambda_2)((1 - w_3)\beta_3 S_m) + \lambda_2 \mu_m + (\lambda_3 - \lambda_4)((1 - w_1)\beta_1 S_h) \\
\frac{-d\lambda_3}{dt} &= \frac{\partial H}{\partial S_h} = -A_1 - \lambda_3 A + (\lambda_3 - \lambda_4)((\beta_1 I_m + \beta_2 I_h)(1 - w_1)) + \lambda_3 \mu_h \\
\frac{-d\lambda_4}{dt} &= \frac{\partial H}{\partial I_h} = -A_2 + (\lambda_3 - \lambda_4)(\beta_2 S_h (1 - w_2)) + \lambda_4(\phi + \mu_h + \delta_1) + (\lambda_4 - \lambda_5)(1 - w_2)\alpha - \lambda_6 \phi \\
\frac{-d\lambda_5}{dt} &= \frac{\partial H}{\partial Q_h} = \gamma(\lambda_5 - \lambda_6) + \lambda_5(\delta_2 + \mu_h) \\
\frac{-d\lambda_6}{dt} &= \frac{\partial H}{\partial R_h} = \varepsilon(\lambda_6 - \lambda_3) + \lambda_6 \mu_h
\end{aligned}\right\} \tag{12}$$

With the tranversality conditions

$$\lambda_1(t_f) = \lambda_2(t_f) = \lambda_3(t_f) = \lambda_4(t_f) = \lambda_5(t_f) = \lambda_6(t_f) = 0 \tag{13}$$

On the interior of the control set, where $0 \le w_i \le 1$, for $i = 1,2,3$, we have

$$0 = \frac{\partial H}{\partial w_1} = a_1 w_1 - \lambda_3(\beta_1 I_m S_h + \beta_2 I_h S_h) + \lambda_4(\beta_1 I_m S_h + \beta_2 I_h + S_h) \tag{14}$$

$$0 = a_1 w_1 - (\lambda_4 - \lambda 3)[\beta_1 I_m S_h + \beta_2 I_h S_h]$$

$$w_1^* = \frac{(\lambda_4 - \lambda_3)(\beta_1 I_m^* S_h^* + \beta_2 I_h^* S_h^*)}{a_1} \qquad (15)$$

Similarly

$$w_2^* = \alpha I_h^* (\frac{\lambda_5 - \lambda_4}{a_2}) \qquad (16)$$

$$w_3^* = \frac{(\lambda_2 - \lambda_1)\beta_3 S_m^* I*_m}{a_3} \qquad (17)$$

**4. Numerical Solution**

In this part, numerical analysis of the quarantine Lassa fever Model is examined with study controls affecting the transmission dynamics. The optimality system is solved to obtained the set of optimal control. The system which referred to as optimality comprises of the state and adjoint systems is evaluated by an iterative scheme, first with the state equation with guess to for the controls over a period of time using Runge-Kutta scheme of order 4-5. owning to the transversality conditions in (11), backward forth order Runge-Kutta scheme is used to evaluate the adjoint equations by using current iterations, then the value from the characteristion. This process is repeatedly performed and stopped when the current values corresponds to the previous unknowns, (Agusto *et. al*, 2012)[2]. Using the following initial conditions and the set of weight factors according to (Momoh and Fügenschuh 2016)[7], $S_h(0) = 500$ , $S_m(0) = 2000$ , $I_h(0) = 10$ , $I_m(0) = 100$ , $Q_h(0) = 5$ , $R_h(0) = 10$ $A_1$ =60, $A_2$ =500, $A_3$ =60, $a_1$ =25, $a_2$ =20, and $a_3$ =30.

Table 1: **State variables and Descriptions of the Lassa fever Quarantine Virus Model**

| Variable | Description |
|---|---|

| | |
|---|---|
| $S_m$ | Susceptible Rats at a given time t. |
| $I_m$ | Infected Rats at a given time t. |
| $S_h$ | Susceptible humans a given time t. |
| $I_h$ | Infected humans at a given time t. |
| $Q_h$ | Quarantined/ isolated humans at a given time t. |
| $R_h(t)$ | Number of Recovered humans at a given time t. |

Table 2: **The baseline Parameter values for the model analysis**

| Parameters | Description | Value | References |
|---|---|---|---|
| $\theta$, | Recruitment rates of infective immigrants rats and susceptible rats receptively | 0.4 | (Momoh , 2016)[7] |
| $\varepsilon$ | The progression rate from the recovered class to Susceptible class due to loss of immunity. | 0.012 | (Abdullahi *et al.*, 2015)[1] |
| $w_1$ | Control measures through Public Education of the mode spread of Lassa fever Virus | 0.5 | Assumed |
| $w_2$ | Control measures by adopting proper standard precautions in treatment of Lassa fever Virus. | 0.3 | Assumed |
| $w_3$ | Control measures by proper hygiene and environmental sanitation. | 0.3 | Assumed |
| $\alpha$ | The progression rate from the infective class the Quarantine class | 0.0025 | (Abdullahi *et al.*, 2015)[1] |
| $\phi$ | Progression rate from infected class to recovered Class due to early treatment | 0.009 | Assumed |
| $\beta_1$ | Force of infection between infected Rats and Susceptible human. | 0.8 | (Abdullahi *et al.*, 2015)[1] |
| $\beta_2$ | Force of infection between Susceptible human and infected human. | 0.8 | (Abdullahi *et al.*, 2015)[1] |
| $\beta_3$ | Force of infection between susceptible Rats and infected Rats. | 0.0714 | (Abdullahi *et al.*, 2015)[1] |
| $\mu_m$ | Natural death rate of Rats. | 0.0038 | (Abdullahi *et al.*, 2015)[1] |
| $\mu_H$ | Natural death rate of Humans. | 0.005 | (Abdullahi *et al.*, 2015)[1] |
| $\delta_1$ | Induced death due to infection. | 0.01626 | (NCDC, 2017)[9] |
| $\delta_2$ | Induced death due to poor Medical services. | 0.00321 | (NCDC, 2017)[9] |
| $\Lambda\ \pi_m$ | Recruitment rates of Humans and Rats | 0.038, 0.56 | Assumed |

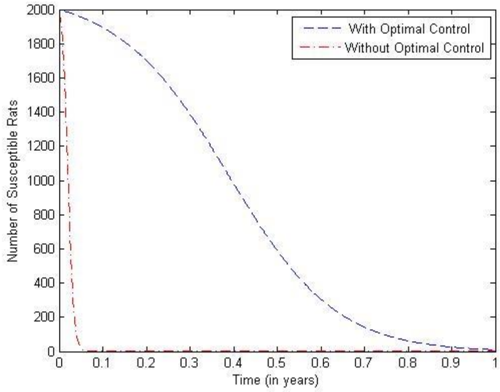


*Fig. 1* Graph of Susceptible Rat population

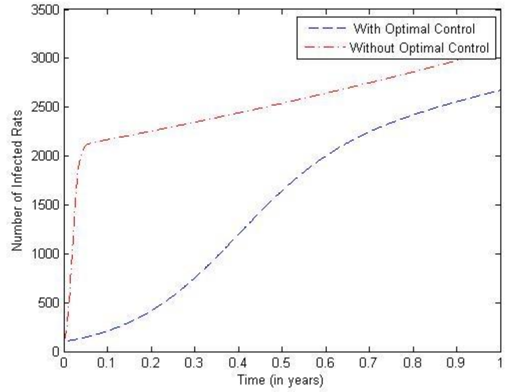


*Fig. 2* Graph of the Infected Rat Population

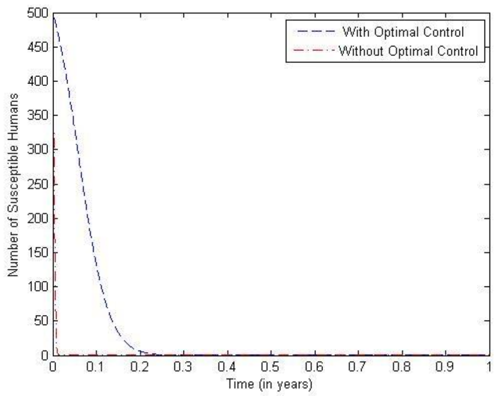


*Fig. 3* Graph of Susceptible Human Population

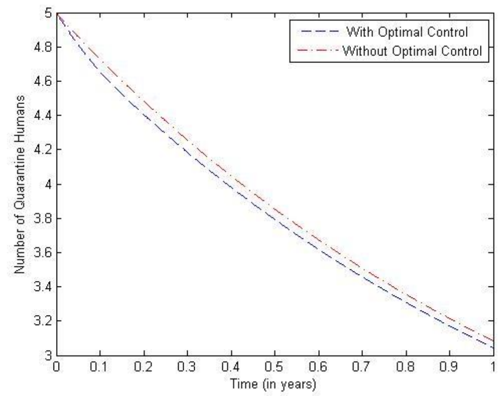


*Fig. 4* Graph of Quarantine Human

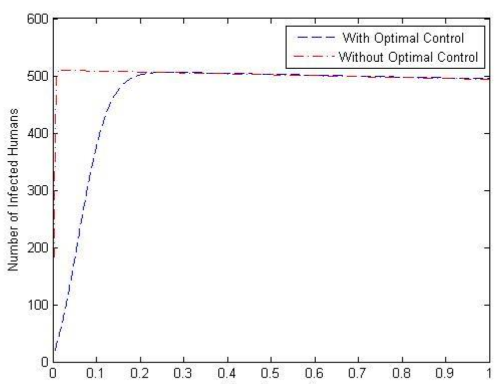


*Fig. 5* Graph of Infected Human

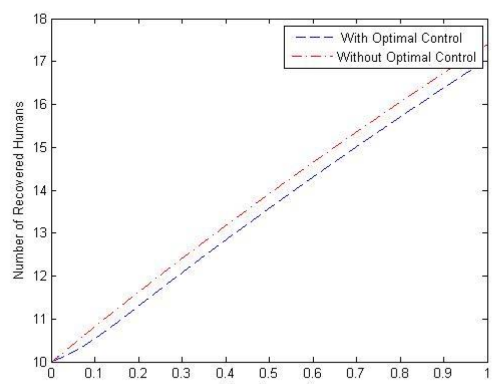


*Fig. 6* Graph of Recovered Human Population

**Fig. 1: Graph of the Susceptible Rat Population**

Without control strategy implemented, the rate of infection of the mastomys is astronomical as the graph indicated. More susceptible mastomys are infected which depleted rapidly the susceptible class and make available more infected rats whcih can be a menace to the human poplulation. But as control is implemented, the susceptible class is steadily maintained and the reduction is due to the elimination of the mastomys rather than being infected. The environmental and sanitation control strategies implemented help to reduce the number of infected rats which in turn lead to reduction in the spread of the infection within human population.

**Fig. 2: Graph of the Infected Rat Population**

The graph above indicated what happens with the model with and without control strategy. The population of the infected rats decreases with increase in the control parameter $w_3$ while the population of the infected rats increases in the absence of no control. The more effective the control strategy (as $w_3 \to 1$), the lesser the infected rats and the more stable the disease free equilibrium among the human.

**Fig. 3: Graph showing Susceptible Human Population**

The graph 3 indicated changes in susceptible human population with and without the control. Without control, the disease become endemic and more susceptible human are infected while with control, the susceptible compartment is steady. The graph without control also indicated that the disease can affect an entire population in less than 1 months if proper care is not taken. This indicated that proper and adequate caution must be taken when an infected individual (or suspected to be infected) is discover within the population so that the entire community can be prevented from the break-out of the infection.

**Fig. 4: Graph showing infected human Population**

Infected human increases without bound in the absence of control. This confirmed that the infection is highly infectious when nothing is done to curtail it. With the control strategy in place, the rate of infection becomes reduced. However, the menace still persist in the population due to contact with already infected individuals. Thus, there is the need to intensify the awareness about the spread of the infection together with quarantining the infected ones so that the disease burden can be lowered.

**Fig. 5: Graph showing Quarantine Human Population**

Numbers of quarantine cases for both models (with and without control) reduces due to neglect or other factor. It is expected that once more cases of infection is discovered, more cases should be quarantine to curtail the spread. However, due to improper education or attitudinal problem of some people, they tend to hid an infected individual from being taken to custody for treatment and this affect the rate of change in population of the infected as more people come into contact the disease.

**Fig. 6: Graph showing Recovered Human population**

The treatment compartment increased due to the fact that effective treatment exist for this deadly infection. However, the number of recovered individual is lower compared with the number of infected, which means that some are avoiding going for treatment due to the fact that the cost is high for the masses (the most affected people). Thus, there's the need for government to intervene in improving the standard of living of the masses so that they can be well equipped to take care of themselves and environment.

## 5. Conclusion

In this paper, we presented a Lassa fever quarantine model using a deterministic system of differential equations and established the existence and necessary condition for optimal control of the disease. $w_1$, $w_2$ and $w_3$ represented public education on the mode of Lassa fever spread, adopting proper standard precautions in treatment of Lassa fever virus and proper hygiene with environmental sanitation as control measures. The effects of these control strategies on model were shown in graphical representation.

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