*Journal of the Nigerian Association of Mathematical Physics*
*Vol.36(2), 2016 pp25-28*


# Application of Matrices to the Theory of Games

**Orverem J.M. and Azuaba E.**

Department of Mathematical Sciences and Information Technology
Federal University P.M.B. 5001, Dutsinma, Nigeria.

Department of Mathematics Federal University of Technology
Minna,Nigeria

orveremjoel@yahoo.com, 07034848671

emmanuelazu2012@yahoo.com, 08135819787

## Abstract

*In this paper, formation of payoff matrix of games and evaluation of saddle points of games is obtained. In addition, determination of strategies used by players in a game as well as the expected payoff of a player is obtained.*

**Key Words:***Matrices, Theory of Games, Saddle Points, Payoff of a Player, Strategies of Players.*

## 1. Introduction

A game is a competitive situation in which each of a number of players is pursuing his objective in direct conflict with the other players [1]. This paper is limited to games played by two players, usually denoted by $R$ and $C$. It is assumed that, player $R$ has $m$ possible moves and that, player $C$ has $n$ moves. An $m \times n$ matrix is formed by labeling it rows, from top to bottom, with the moves of $R$, and labeling it columns, from left to right, with the moves of $C$.

An agent playing a game is called a player [2]. In an $m \times n$ matrix, the entry $a_{ij}$ in row $i$ and column $j$, which indicates the amount (money or some other valuable item) received by $R$ if player $R$ makes his $ith$ move and player $C$ make his $jth$ move is called a payoff [3]. A table that shows the payoff for every possible action by each player is called a payoff matrix.

Games of strategy are games which require skills on the part of the players, the outcomes and winnings are determined by the skills of the players [4]. Examples are chess, checkers, bridge, nim and poker. A strategy for a player is a decision for choosing his moves. If the payoff matrix of a matrix game contains an entry $a_{rs}$, which is at the same time the minimum of row $r$ and the maximum of column $s$, then $a_{rs}$ is called a saddle point or the value of the game [5].

## 2. Formation of payoff matrix of games

Consider a two person, constant-sum game with the $m \times n$ payoff matrix $A = (a_{ij})$, so that player $R$ has $m$ moves and player $C$ has $n$ moves. If player $R$ plays his $ith$ move, he is assured of winning at least the smallest entry in the $ith$ row of $A$, no matter what $C$ does. Thus $R$'sbest course of action is to choose that move which will maximize his assured winnings in spite of $C$'sbest counter move. Player $R$ will get his largest payoff by maximizing his smallest gain. Player $C$'sgoals are in direct conflict with those of player $R$: he is trying to keep $R$'s winning to a minimum. If $C$ plays his $jth$ move, he is assured of losing no more than the largest entry in the $jth$ column of $A$, no matter what $R$ does. Thus $C$'s best course of action is to choose that move which will minimize his assured losses in spite of $R$'s best counter move. Player $C$ will do his best by minimizing his largest loss.

**Example 2.1:** Consider the game of matching pennies, consisting of two players, $R$ and $C$, each of whom has a penny in his hand. Each player shows one side of the coin without knowing his opponent's choice. If both players are showing the same side of the coin, then$R$ wins ₦1 from $C$, otherwise, $C$ wins ₦1 from $R$. In this two-person, zero-sum game, each player has two possible moves, he can show a tail or he can show a head. The payoff matrix is given as

C

H T

$$\text{R} \quad \begin{matrix} H \\ T \end{matrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

From example 2.1 above, $H$ and $T$ represent the head and tail respectively of the penny or coin.

**Example 2.2:** There are two supplies, firms $R$ and $C$, of a new specialized type of tire that has 100,000 customers. Each company can advertise its product on TV or in the newspapers. A Marketing firm determines that if both firms advertise on TV, then firm $R$ gets 40,000 customers and firm $C$ gets 60,000 customers. If they both use newspapers, then each gets 50,000 customers. If $R$ uses newspapers and $C$ uses TV, then $R$ gets 60,000 customers. If $R$ uses TV and $C$ uses newspapers, they each get 50,000 customers. We can consider this situation as a game between firms $R$ and $C$, with payoff matrix as

Firm C

TV Newspapers

$$\text{Firm R} \quad \begin{matrix} TV \\ Newspapers \end{matrix}\begin{pmatrix} 40{,}000 & 50{,}000 \\ 60{,}000 & 50{,}000 \end{pmatrix}.$$

From example 2.2 above, the entries in the matrix indicate the number of customers secured by firm $R$.

**Example 2.3:** Firms $A$ and $B$, both handling specialized sporting equipment, are planning to locate in either Chito or Z/Biam. If they both locate in the same town, each will capture 50 percent of the trade. If $A$ locates in Chito and $B$ locates in Z/Biam, then $A$ will capture 60 percent of the business (and $B$ will keep 40 percent); if $A$ locates in Z/Biam and $B$ locates in Chito, then $A$ will hold on to 25 percent of the business (and $B$ to 75 percent). The payoff matrix for the game is

Firm B

Chito Z/Biam

$$\text{Firm A} \begin{matrix} Chito \\ Z/Biam \end{matrix} \begin{pmatrix} 50 & 60 \\ 25 & 50 \end{pmatrix}.$$

## 3. Evaluation of saddle points of games

If the payoff of a matrix game contains an entry $a_{rs}$, which is at the same time the minimum of row $r$ and the maximum of column $s$, then $a_{rs}$ is called a saddle point. Also, $a_{rs}$ is called the value of the game. If the value of a zero-sum game is zero, the game is said to be fair.

If $a_{rs}$ is a saddle point for a matrix game, then player $R$ will be assured of winning at least $a_{rs}$ by playing his $rth$ move and player $C$ will be guaranteed that he will lose no more than $a_{rs}$ by playing his $sth$ move. This is the best that each player can do.

**Example 3.1:** Consider a game with payoff matrix

C

$$R \begin{pmatrix} 0 & -3 & -1 & 3 \\ 3 & 2 & 2 & 4 \\ 1 & 4 & 0 & 6 \end{pmatrix}.$$

To determine whether this game has a saddle point, we write the minimum of each row to the right of the row and the maximum of each column at the bottom of each column. Thus, we have as follows:

CRow minima

$$R \begin{pmatrix} 0 & -3 & -1 & 3 \\ 3 & 2 & 2 & 4 \\ 1 & 4 & 0 & 6 \end{pmatrix} \begin{matrix} -3 \\ 2 \\ 0 \end{matrix}$$

Column maxima3 4 26

Entry $a_{23} = 2$ is both the least entry in the second row and the largest entry in the third column. Hence, it is a saddle point for the game, which is then a strictly determined game. The value of the game is 2 and player $R$ has an advantage. The best course of action for $R$ is to play his second move, he will win at least 2 units from $C$, no matter what $C$ does. The best course of action for $C$ is to play his third move, he will limit his loss to not more than 2 units, no matter what $R$ does.

**Example 3.2:** Consider the advertizing game of example 2.2 (section 2). The payoff matrix is:

Firm C

TV Newspapers Row minima

$$\text{Firm R} \quad \begin{matrix} TV \\ Newspapers \end{matrix} \begin{pmatrix} 40{,}000 & 50{,}000 \\ 60{,}000 & 50{,}000 \end{pmatrix} \begin{matrix} 40{,}000 \\ 50{,}000 \end{matrix}.$$

Column maxima 60,000 50,000

Thus entry $a_{22} = 50{,}000$ is a saddle point. The best course of action for both firms is to advertise in newspapers. The game is strictly determined with the value 50,000.

A game may have more than one saddle point. However, it can be proved that all the saddle points must have the same value as can be seen in the example below.

**Example 3.3:** Find all saddle points for the following matrix game.

$$\begin{pmatrix} 5 & 2 & 4 & 2 \\ 0 & -1 & 2 & 0 \\ 3 & 2 & 3 & 2 \\ 1 & 0 & -1 & -1 \end{pmatrix}$$

Consider the game as been played by two players $R$ and $C$ respectively. We can as usual write the minimum of each row to the right of the row and the maximum of each column at the bottom of each column, so as to determine the saddle points.

C Row minima

$$R \begin{pmatrix} 5 & 2 & 4 & 2 \\ 0 & -1 & 2 & 0 \\ 3 & 2 & 3 & 2 \\ 1 & 0 & -1 & -1 \end{pmatrix} \begin{matrix} 2 \\ -1 \\ 2 \\ -1 \end{matrix}$$

Column maxima 5 2 4 2

Entries $a_{12}, a_{14}, a_{32}$ and $a_{34}$ are all saddle points and have the same value, 2. They appear shaded in the payoff matrix. The value of the game is also 2.

It is important to note that, there are many games that are not strictly determined.

**Example 3.4:** Consider the game with payoff matrix

C Row minima

$$R \quad \begin{pmatrix} 1 & 6 & -1 \\ 3 & -2 & 4 \\ 4 & 5 & -3 \end{pmatrix} \begin{matrix} -1 \\ -2. \\ -3 \end{matrix}$$

Column maxima 4 6 4

It is clear that there is no saddle point hence, the game is not strictly determined.

In the penny-matching game of example 2.1 in section 2, it is clear that this game is not strictly determined, because it has no saddle point.

## 4. Determination of strategies used by players in a game and the expected payoff

A strategy for a player is a decision for choosing his moves. Consider now the penny-matching game in example 2.1 section 2. Suppose that, in the repeated play of the game, player $R$ always chooses the first row (he chooses to show heads), in the hope that payer $C$ will always choose the first column (play heads), thereby ensuring a win of ₦1 for himself. However, as player $C$ begins to notice that player $R$ always chooses his first row, then player $C$ will chooses his second column, resulting in a loss of ₦1 for $R$. Similarly, if $R$ always chooses the second row, then $C$ will choose the first column, resulting in a loss of ₦1 for $R$. We can thus conclude that each player must somehow keep the other player from anticipating his choice of moves.

Suppose that we have a matrix game with an $m \times n$ payoff matrix $A$. Let $p_i, 1 \leq i \leq m$, be the probability that $R$ chooses the $ith$ row of $A$ (that is, chooses his $ith$ move). Let $q_j$, $1 \leq j \leq n$, be the probability that $C$ chooses the $jth$ column of $A$. The vector $P = (p_1 \quad p_2 \quad \cdots \quad p_m)$ is called a strategy for player $R$. The strategy for player $C$ is given by the vector

$$Q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}.$$

Of course, the probabilities $p_i$ and $q_j$ in this definition satisfy

$$p_1 + p_2 + \cdots + p_m = 1$$

$$q_1 + q_2 + \cdots + q_n = 1.$$

If a matrix game is strictly determined, then optimal strategies for $R$ and $C$ are strategies having 1 as one component and zero for all other components. Such strategies are called pure strategies. A strategy that is not pure is called a mixed strategy.

**Example 4.1:** Consider a game with payoff matrix

$$\begin{array}{cc} & C \\ R & \begin{pmatrix} 0 & -3 & -1 & 3 \\ 3 & 2 & 2 & 4 \\ 1 & 4 & 0 & 6 \end{pmatrix}. \end{array}$$

We can find the pure strategy for player $R$ and the pure strategy for player $C$ as follows

C Row minima

$$R \begin{pmatrix} 0 & -3 & -1 & 3 \\ 3 & 2 & 2 & 4 \\ 1 & 4 & 0 & 6 \end{pmatrix} \begin{matrix} -3 \\ 2. \\ 0 \end{matrix}$$

Column maxima 3 4 2 6

The saddle point of the game is the entry $a_{23} = 2$ and the game is strictly determined. The pure strategy for player $R$ is $P = (0 \quad 1 \quad 0)$ and the pure strategy for player $C$ is

$$Q = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Now consider a matrix game with payoff matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Suppose that $P = (p_1 \quad p_2)$ and $Q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ are strategies for $R$ and $C$, respectively. Then if $R$ plays his first row with probability $p_1$ and $C$ plays his first column with probability $q_1$, then $R$'s expected payoff is $p_1 q_1 a_{11}$. Similarly, the remaining three possibilities is shown in the table 4.0 below. The expected payoff $E(P, Q)$ of the game to player $R$ is then the sum of the four quantities in the right most column. We obtain; $E(P, Q) = p_1 q_1 a_{11} + p_1 q_2 a_{12} + p_2 q_1 a_{21} + p_2 q_2 a_{22}$, which can be written in matrix form as $E(P, Q) = PAQ$.

Moves

| Player $R$ | Player $C$ | Probability | Payoff to Player $R$ | Expected payoff to Player $R$ |
|---|---|---|---|---|
| Row 1 | Column 1 | $p_1q_1$ | $a_{11}$ | $p_1q_1a_{11}$ |
| Row 1 | Column2 | $p_1q_2$ | $a_{12}$ | $p_1q_2a_{12}$ |
| Row 2 | Column 1 | $p_2q_1$ | $a_{21}$ | $p_2q_1a_{21}$ |
| Row 2 | Column 2 | $p_2q_2$ | $a_{22}$ | $p_2q_2a_{22}$ |

**Table 4.0: Table showing the payoff and expected payoff to Player*R*.**

The same analysis applies to a matrix game with an $m \times n$ matrix$A$. Thus, if $P = (p_1 \quad p_2 \quad \cdots \quad p_m)$ and $Q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$ are strategies for player $R$ and $C$, respectively, then the expected payoff to player $R$ is given by $E(P,Q) = PAQ$.

**Example 4.2:** Consider a matrix game with payoff matrix

$$A = \begin{pmatrix} 2 & -2 & 3 \\ 4 & 0 & -3 \end{pmatrix}.$$

If $P = (1/4 \quad 3/4)$and $Q = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$ are strategies for $R$ and $C$, respectively, then the expected payoff to $R$ is

$$E(P,Q) = (1/4 \quad 3/4)\begin{pmatrix} 2 & -2 & 3 \\ 4 & 0 & -3 \end{pmatrix}\begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} = 3/6 = 1/2.$$

Therefore, the expected payoff to $R$ is $1/2$.

If $P = (3/4 \quad 1/4)$ and $Q = \begin{pmatrix} 1/3 \\ 2/3 \\ 0 \end{pmatrix}$ are strategies for $R$ and $C$, respectively, then using the same payoff matrix in example 4.2 above, we find that the expected payoff to $R$ is $-1/6$.

Thus in the first case, $R$ gains $1/2$ from $C$ whereas in the second case, $R$ loses $1/6$ to $C$.

## 5. Conclusion

In as much as games are represented by game trees, matrices are also used to represent games, which is the basis of this research paper. Matrices enable us to find the saddle points of a game and as well, the expected payoff of a player.

## References

[1] Bernard K. (1980). Introductory Linear Algebra with Application 2nd edition, United States of America: Macmillan Publishing Co.

[2] Stephen, R. (1985). An introduction to linear Algebra with Applications, United States of America: Macmillan Publishing Co.

[3] Herstein, I. N. and David J. W. (1988). Matrix Theory and Linear Algebra, United States of America: Macmillan Publishing Company a division of Macmillan, Inc.

[4] Luce, R. D. and Howard R. (1957). Games and decisions, New York: John Wiley and Sons, Inc.

[5] Oborne, M. J. (2004). An introduction to Game theory, New York: Oxford University Press.